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Basic Facts about Inaccessible and Measurable Cardinals

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Summary. Inaccessible, strongly inaccessible and measurable cardinals are defined, and it is proved that a measurable cardinal is strongly inaccessible. Filters on sets are defined, some facts related to the section about cardinals are proved. Existence of the Ulam matrix on non-limit cardinals is proved.

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The articles [13], [10], [14], [15], [8], [7], [12], [11], [2], [9], [3], [1], [4], [5], and [6] provide the notation and terminology for this paper.

1. Some Facts about Filters and Ideals on Sets

Let us note that there exists an ordinal number which is limit.

Let X, Y be sets. Then $X \setminus Y$ is a subset of X.

We now state the proposition

(1) For every set x and for every infinite set X holds $\overline{\overline{\{x\}}} < \overline{\overline{X}}$.

Let X be an infinite set. One can check that $\overline{\overline{X}}$ is infinite.

The scheme *ElemProp* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters meet the following requirement:

- $\mathcal{B} \in \{y; y$ ranges over elements of $\mathcal{A} : \mathcal{P}[y]\}$.

For simplicity, we use the following convention: N is a cardinal number, M is an aleph, X is a non empty set, Y, Z, Z_1, Z_2, Y_1, Y_2 are subsets of X, and S is a family of subsets of X.

Next we state the proposition

(2)(i) $\{X\}$ is a non empty family of subsets of X,

(ii) $\emptyset \notin \{X\}$, and

(iii) for all Y_1, Y_2 holds if $Y_1 \in \{X\}$ and $Y_2 \in \{X\}$, then $Y_1 \cap Y_2 \in \{X\}$ and if $Y_1 \in \{X\}$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \{X\}$.

Let us consider X. A non empty family of subsets of X is said to be a filter of X if:

(Def. 1) $\emptyset \notin$ it and for all Y_1, Y_2 holds if $Y_1 \in$ it and $Y_2 \in$ it, then $Y_1 \cap Y_2 \in$ it and if $Y_1 \in$ it and $Y_1 \subseteq Y_2$, then $Y_2 \in$ it.

Next we state two propositions:

(3) Let F be a set. Then F is a filter of X if and only if the following conditions are satisfied:

(i) F is a non empty family of subsets of X,

(ii) $\emptyset \notin F$, and

(iii) for all Y_1, Y_2 holds if $Y_1 \in F$ and $Y_2 \in F$, then $Y_1 \cap Y_2 \in F$ and if $Y_1 \in F$ and $Y_1 \subseteq Y_2$, then $Y_2 \in F$.

(4) $\{X\}$ is a filter of X.

In the sequel F, U_1 denote filters of X.
One can prove the following propositions:

(5) $X \in F$.

(6) If $Y \in F$, then $X \setminus Y \notin F$.

(7) Let I be a non empty subset of 2^X. Suppose that for every Y holds $Y \in I$ iff $Y^c \in F$. Then $X \notin I$ and for all Y_1, Y_2 holds if $Y_1 \in I$ and $Y_2 \in I$, then $Y_1 \cup Y_2 \in I$ and if $Y_1 \in I$ and $Y_2 \subseteq Y_1$, then $Y_2 \in I$.

Let us consider X, S. We introduce $\operatorname{dual} S$ as a synonym of S^c.
In the sequel S is a non empty family of subsets of X.
Let us consider X, S. One can verify that S^c is non empty.
The following two propositions are true:

(8) $\operatorname{dual} S = \{Y : Y^c \in S\}$.

(9) $\operatorname{dual} S = \{Y^c : Y \in S\}$.

Let us consider X. A non empty family of subsets of X is said to be an ideal of X if:

(Def. 2) $X \notin$ it and for all Y_1, Y_2 holds if $Y_1 \in$ it and $Y_2 \in$ it, then $Y_1 \cup Y_2 \in$ it and if $Y_1 \in$ it and $Y_2 \subseteq Y_1$, then $Y_2 \in$ it.

Let us consider X, F. Then $\operatorname{dual} F$ is an ideal of X.
In the sequel I denotes an ideal of X.
We now state two propositions:

(10) For every Y holds $Y \notin F$ or $Y \notin \operatorname{dual} F$ and for every Y holds $Y \notin I$ or $Y \notin \operatorname{dual} I$.

(11) $\emptyset \in I$.

Let us consider X, N, S. We say that S is multiplicative with N if and only if:

(Def. 3) For every non empty set S_1 such that $S_1 \subseteq S$ and $\overline{\overline{S_1}} < N$ holds $\bigcap S_1 \in S$.

Let us consider X, N, S. We say that S is additive with N if and only if:

(Def. 4) For every non empty set S_1 such that $S_1 \subseteq S$ and $\overline{\overline{S_1}} < N$ holds $\bigcup S_1 \in S$.

Let us consider X, N, F. We introduce F is complete with N as a synonym of F is multiplicative with N.

Let us consider X, N, I. We introduce I is complete with N as a synonym of I is additive with N.

One can prove the following proposition

(12) If S is multiplicative with N, then $\operatorname{dual} S$ is additive with N.

Let us consider X, F. We say that F is uniform if and only if:

(Def. 5) For every Y such that $Y \in F$ holds $\overline{\overline{Y}} = \overline{\overline{X}}$.

We say that F is principal if and only if:

(Def. 6) There exists Y such that $Y \in F$ and for every Z such that $Z \in F$ holds $Y \subseteq Z$.

We say that F is an ultrafilter if and only if:

(Def. 7) For every Y holds $Y \in F$ or $X \setminus Y \in F$.

Let us consider X, F, Z. The functor Extend_Filter(F,Z) yielding a non empty family of subsets of X is defined as follows:

(Def. 8) Extend_Filter$(F,Z) = \{Y : \bigvee_{Y_2} (Y_2 \in \{Y_1 \cap Z : Y_1 \in F\} \wedge Y_2 \subseteq Y)\}$.

The following propositions are true:

(13) For every Z_1 holds $Z_1 \in$ Extend_Filter(F,Z) iff there exists Z_2 such that $Z_2 \in F$ and $Z_2 \cap Z \subseteq Z_1$.

(14) If for every Y_1 such that $Y_1 \in F$ holds Y_1 meets Z, then $Z \in$ Extend_Filter(F,Z) and Extend_Filter(F,Z) is a filter of X and $F \subseteq$ Extend_Filter(F,Z).

Let us consider X. The functor Filters$X$ yields a non empty family of subsets of 2^X and is defined as follows:

(Def. 9) Filters$X = \{S; S$ ranges over subsets of 2^X: S is a filter of $X\}$.

One can prove the following proposition

(15) For every set S holds $S \in$ Filters$X$ iff S is a filter of X.

In the sequel F_1 denotes a non empty subset of Filters$X$.
The following two propositions are true:

(16) If F_1 is $\subseteq$-linear, then $\bigcup F_1$ is a filter of X.

(17) For every F there exists U_1 such that $F \subseteq U_1$ and U_1 is an ultrafilter.

In the sequel X is an infinite set, Y is a subset of X, and F, U_1 are filters of X.
Let us consider X. The functor Frechet_Filter$X$ yielding a filter of X is defined by:

(Def. 10) Frechet_Filter$X = \{Y : \overline{\overline{X \setminus Y}} < \overline{\overline{X}}\}$.

Let us consider X. The functor Frechet_Ideal$X$ yielding an ideal of X is defined by:

(Def. 11) Frechet_Ideal$X$ = dual Frechet_Filter$X$.

One can prove the following four propositions:

(18) $Y \in$ Frechet_Filter$X$ iff $\overline{\overline{X \setminus Y}} < \overline{\overline{X}}$.

(19) $Y \in$ Frechet_Ideal$X$ iff $\overline{\overline{Y}} < \overline{\overline{X}}$.

(20) If Frechet_Filter$X \subseteq F$, then F is uniform.

(21) If U_1 is uniform and an ultrafilter, then Frechet_Filter$X \subseteq U_1$.

Let us consider X. Observe that there exists a filter of X which is non principal and an ultrafilter.
Let us consider X. Observe that every filter of X which is uniform and an ultrafilter is also non principal.
We now state two propositions:

(22) For every an ultrafilter filter F of X and for every Y holds $Y \in F$ iff $Y \notin$ dual F.

(23) If F is non principal and an ultrafilter and F is complete with $\overline{\overline{X}}$, then F is uniform.

2. Inaccessible and Measurable Cardinals, Ulam Matrix

Next we state the proposition

(24) $N^+ \leq 2^N$.

We say that Generalized Continuum Hypothesis holds if and only if:

(Def. 12) For every aleph N holds $N^+ = 2^N$.

Let I_1 be an aleph. We say that I_1 is inaccessible if and only if:

(Def. 13) I_1 is regular and limit.

We introduce I_1 is inaccessible cardinal as a synonym of I_1 is inaccessible.
Let us observe that every aleph which is inaccessible is also regular and limit.
One can prove the following proposition

(25) $\aleph_0$ is inaccessible.

Let I_1 be an aleph. We say that I_1 is strong limit if and only if:

(Def. 14) For every N such that $N < I_1$ holds $2^N < I_1$.

We introduce I_1 is strong limit cardinal as a synonym of I_1 is strong limit.
Next we state two propositions:

(26) $\aleph_0$ is strong limit.

(27) If M is strong limit, then M is limit.

Let us note that every aleph which is strong limit is also limit.
We now state the proposition

(28) If Generalized Continuum Hypothesis holds, then if M is limit, then M is strong limit.

Let I_1 be an aleph. We say that I_1 is strongly inaccessible if and only if:

(Def. 15) I_1 is regular and strong limit.

We introduce I_1 is strongly inaccessible cardinal as a synonym of I_1 is strongly inaccessible.
One can check that every aleph which is strongly inaccessible is also regular and strong limit.
Next we state two propositions:

(29) $\aleph_0$ is strongly inaccessible.

(30) If M is strongly inaccessible, then M is inaccessible.

Let us observe that every aleph which is strongly inaccessible is also inaccessible.
Next we state the proposition

(31) If Generalized Continuum Hypothesis holds, then if M is inaccessible, then M is strongly inaccessible.

Let us consider M. We say that M is measurable if and only if:

(Def. 16) There exists a filter U_1 of M such that U_1 is complete with M and U_1 is non principal and an ultrafilter.

We introduce M is measurable cardinal as a synonym of M is measurable.
We now state two propositions:

(32) For every limit ordinal number A and for every set X such that $X \subseteq A$ holds if $\sup X = A$, then $\bigcup X = A$.

(33) If M is measurable, then M is regular.

Let us consider M. One can check that M^+ is non limit.
Let us note that there exists a cardinal number which is non limit and infinite.
Let us note that every aleph which is non limit is also regular.
Let M be a non limit cardinal number. The functor predecessor$M$ yielding a cardinal number is defined as follows:

(Def. 17) $M = (\text{predecessor} M)^+$.

Let M be a non limit aleph. Note that predecessor$M$ is infinite.
Let X be a set and let N, N_1 be cardinal numbers. An Inf Matrix of N, N_1, X is a function from $[:N, N_1:]$ into X.
For simplicity, we follow the rules: X is a set, M is a non limit aleph, F is a filter of M, N_1, N_2 are elements of predecessor$M$, K_1, K_2 are elements of M, and T is an Inf Matrix of predecessor$M$, M, 2^M.
Let us consider M, T. We say that T is Ulam Matrix of M if and only if the conditions (Def. 18) are satisfied.

(Def. 18)(i) For all N_1, K_1, K_2 such that $K_1 \neq K_2$ holds $T(N_1, K_1) \cap T(N_1, K_2)$ is empty,

(ii) for all K_1, N_1, N_2 such that $N_1 \neq N_2$ holds $T(N_1, K_1) \cap T(N_2, K_1)$ is empty,

(iii) for every N_1 holds $\overline{\overline{M \setminus \bigcup\{T(N_1, K_1) : K_1 \in M\}}} \leq \text{predecessor} M$, and

(iv) for every K_1 holds $\overline{\overline{M \setminus \bigcup\{T(N_1, K_1) : N_1 \in \text{predecessor} M\}}} \leq \text{predecessor} M$.

Next we state four propositions:

(34) There exists T such that T is Ulam Matrix of M.

(35) Let given M and I be an ideal of M. Suppose I is complete with M and Frechet_Ideal$M \subseteq I$. Then there exists a family S of subsets of M such that

(i) $\overline{\overline{S}} = M$,

(ii) for every set X_1 such that $X_1 \in S$ holds $X_1 \notin I$, and

(iii) for all sets X_1, X_2 such that $X_1 \in S$ and $X_2 \in S$ and $X_1 \neq X_2$ holds X_1 misses X_2.

(36) For every X and for every cardinal number N such that $N \leq \overline{\overline{X}}$ there exists a set Y such that $Y \subseteq X$ and $\overline{\overline{Y}} = N$.

(37) For every M it is not true that there exists F such that F is uniform and an ultrafilter and F is complete with M.

In the sequel M is an aleph.
One can prove the following four propositions:

(38) If M is measurable, then M is limit.

(39) If M is measurable, then M is inaccessible.

(40) If M is measurable, then M is strong limit.

(41) If M is measurable, then M is strongly inaccessible.

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