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Cardinal Arithmetics

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Summary. In the article addition, multiplication and power operation of cardinals are introduced. Presented are some properties of equipotence of Cartesian products, basic cardinal arithmetics laws (transformativity, associativity, distributivity), and some facts about finite sets.

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The articles [13], [12], [9], [14], [7], [8], [3], [4], [5], [11], [2], [6], [1], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: A, B are ordinal numbers, K, M, N are cardinal numbers, x, x_1, x_2, y, y_1, y_2, X, Y, Z, X_1, X_2, Y_1, Y_2 are sets, and f is a function.

Next we state several propositions:

(2)[1] $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff there exists f such that $X \subseteq f^\circ Y$.

(3) $\overline{\overline{f^\circ X}} \leq \overline{\overline{X}}$.

(4) If $\overline{\overline{X}} < \overline{\overline{Y}}$, then $Y \setminus X \neq \emptyset$.

(5) If $x \in X$ and $X \approx Y$, then $Y \neq \emptyset$ and there exists x such that $x \in Y$.

(6) $2^X \approx 2^{\overline{\overline{X}}}$ and $\overline{\overline{2^X}} = \overline{\overline{2^{\overline{\overline{X}}}}}$.

(7) If $Z \in Y^X$, then $Z \approx X$ and $\overline{\overline{Z}} = \overline{\overline{X}}$.

Let us consider M, N. The functor $M + N$ yielding a cardinal number is defined as follows:

(Def. 1) $M + N = \overline{\overline{M + N}}$.

Let us observe that the functor $M + N$ is commutative. The functor $M \cdot N$ yields a cardinal number and is defined as follows:

(Def. 2) $M \cdot N = \overline{\overline{[:M, N:]}}$.

Let us observe that the functor $M \cdot N$ is commutative. The functor M^N yields a cardinal number and is defined as follows:

(Def. 3) $M^N = \overline{\overline{M^N}}$.

The following propositions are true:

[1] The proposition (1) has been removed.

(11)[2] $[:X,Y:] \approx [:Y,X:]$ and $\overline{\overline{[:X,Y:]}} = \overline{\overline{[:Y,X:]}}$.

(12) $[:[:X,Y:],Z:] \approx [:X,[:Y,Z:]:]$ and $\overline{\overline{[:[:X,Y:],Z:]}} = \overline{\overline{[:X,[:Y,Z:]:]}}$.

(13) $X \approx [:X,\{x\}:]$ and $\overline{\overline{X}} = \overline{\overline{[:X,\{x\}:]}}$.

(14) $[:X,Y:] \approx [:\overline{\overline{X}},Y:]$ and $[:X,Y:] \approx [:X,\overline{\overline{Y}}:]$ and $[:X,Y:] \approx [:\overline{\overline{X}},\overline{\overline{Y}}:]$ and $\overline{\overline{[:X,Y:]}} = \overline{\overline{[:\overline{\overline{X}},Y:]}}$ and $\overline{\overline{[:X,Y:]}} = \overline{\overline{[:X,\overline{\overline{Y}}:]}}$ and $\overline{\overline{[:X,Y:]}} = \overline{\overline{[:\overline{\overline{X}},\overline{\overline{Y}}:]}}$.

(15) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$, then $[:X_1,X_2:] \approx [:Y_1,Y_2:]$ and $\overline{\overline{[:X_1,X_2:]}} = \overline{\overline{[:Y_1,Y_2:]}}$.

(16) If $x_1 \neq x_2$, then $A+B \approx [:A,\{x_1\}:] \cup [:B,\{x_2\}:]$ and $\overline{\overline{A+B}} = \overline{\overline{[:A,\{x_1\}:] \cup [:B,\{x_2\}:]}}$.

(17) If $x_1 \neq x_2$, then $K+M \approx [:K,\{x_1\}:] \cup [:M,\{x_2\}:]$ and $K+M = \overline{\overline{[:K,\{x_1\}:] \cup [:M,\{x_2\}:]}}$.

(18) $A \cdot B \approx [:A,B:]$ and $\overline{\overline{A \cdot B}} = \overline{\overline{[:A,B:]}}$.

(19) $0 = \overline{\overline{\emptyset}}$ and $1 = \overline{\overline{\mathbf{1}}}$ and $2 = \overline{\overline{\operatorname{succ}\mathbf{1}}}$.

(20) $1 = \mathbf{1}$.

(22)[3] $2 = \{\emptyset, \mathbf{1}\}$ and $2 = \operatorname{succ}\mathbf{1}$.

(23) If $X_1 \approx Y_1$ and $X_2 \approx Y_2$ and $x_1 \neq x_2$ and $y_1 \neq y_2$, then $[:X_1,\{x_1\}:] \cup [:X_2,\{x_2\}:] \approx [:Y_1,\{y_1\}:] \cup [:Y_2,\{y_2\}:]$ and $\overline{\overline{[:X_1,\{x_1\}:] \cup [:X_2,\{x_2\}:]}} = \overline{\overline{[:Y_1,\{y_1\}:] \cup [:Y_2,\{y_2\}:]}}$.

(24) $\overline{\overline{A+B}} = \overline{\overline{A}} + \overline{\overline{B}}$.

(25) $\overline{\overline{A \cdot B}} = \overline{\overline{A}} \cdot \overline{\overline{B}}$.

(26) $[:X,\{0\}:] \cup [:Y,\{1\}:] \approx [:Y,\{0\}:] \cup [:X,\{1\}:]$ and $\overline{\overline{[:X,\{0\}:] \cup [:Y,\{1\}:]}} = \overline{\overline{[:Y,\{0\}:] \cup [:X,\{1\}:]}}$.

(27) $[:X_1,X_2:] \cup [:Y_1,Y_2:] \approx [:X_2,X_1:] \cup [:Y_2,Y_1:]$ and $\overline{\overline{[:X_1,X_2:] \cup [:Y_1,Y_2:]}} = \overline{\overline{[:X_2,X_1:] \cup [:Y_2,Y_1:]}}$.

(28) If $x \neq y$, then $\overline{\overline{X}} + \overline{\overline{Y}} = \overline{\overline{[:X,\{x\}:] \cup [:Y,\{y\}:]}}$.

(29) $M+0=M$.

(31)[4] $(K+M)+N = K+(M+N)$.

(32) $K \cdot 0 = 0$.

(33) $K \cdot 1 = K$.

(35)[5] $(K \cdot M) \cdot N = K \cdot (M \cdot N)$.

(36) $2 \cdot K = K+K$.

(37) $K \cdot (M+N) = K \cdot M + K \cdot N$.

(38) $K^0 = 1$.

(39) If $K \neq 0$, then $0^K = 0$.

(40) $K^1 = K$ and $1^K = 1$.

(41) $K^{M+N} = K^M \cdot K^N$.

[2] The propositions (8)–(10) have been removed.
[3] The proposition (21) has been removed.
[4] The proposition (30) has been removed.
[5] The proposition (34) has been removed.

(42) $(K \cdot M)^N = K^N \cdot M^N$.

(43) $K^{M \cdot N} = (K^M)^N$.

(44) $2^{\overline{\overline{X}}} = \overline{\overline{2^X}}$.

(45) $K^2 = K \cdot K$.

(46) $(K+M)^2 = K \cdot K + 2 \cdot K \cdot M + M \cdot M$.

(47) $\overline{\overline{X \cup Y}} \leq \overline{\overline{X}} + \overline{\overline{Y}}$.

(48) If X misses Y, then $\overline{\overline{X \cup Y}} = \overline{\overline{X}} + \overline{\overline{Y}}$.

In the sequel m, n are natural numbers.
The following propositions are true:

(49) $n + m = n + m$.

(50) $n \cdot m = n \cdot m$.

(51) $\overline{\overline{n+m}} = \overline{\overline{n}} + \overline{\overline{m}}$.

(52) $\overline{\overline{n \cdot m}} = \overline{\overline{n}} \cdot \overline{\overline{m}}$.

(53) For all finite sets X, Y such that X misses Y holds $\mathrm{card}(X \cup Y) = \mathrm{card}\,X + \mathrm{card}\,Y$.

(54) For every finite set X such that $x \notin X$ holds $\mathrm{card}(X \cup \{x\}) = \mathrm{card}\,X + 1$.

(57)[6] For all finite sets X, Y holds $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff $\mathrm{card}\,X \leq \mathrm{card}\,Y$.

(58) For all finite sets X, Y holds $\overline{\overline{X}} < \overline{\overline{Y}}$ iff $\mathrm{card}\,X < \mathrm{card}\,Y$.

(59) For every set X such that $\overline{\overline{X}} = 0$ holds $X = \emptyset$.

(60) For every set X holds $\overline{\overline{X}} = 1$ iff there exists x such that $X = \{x\}$.

(61) For every finite set X holds $X \approx \mathrm{card}\,X$ and $X \approx \overline{\overline{\mathrm{card}\,X}}$ and $X \approx \mathrm{Seg}\,\mathrm{card}\,X$.

(62) For all finite sets X, Y holds $\mathrm{card}(X \cup Y) \leq \mathrm{card}\,X + \mathrm{card}\,Y$.

(63) For all finite sets X, Y such that $Y \subseteq X$ holds $\mathrm{card}(X \setminus Y) = \mathrm{card}\,X - \mathrm{card}\,Y$.

(64) For all finite sets X, Y holds $\mathrm{card}(X \cup Y) = (\mathrm{card}\,X + \mathrm{card}\,Y) - \mathrm{card}(X \cap Y)$.

(65) For all finite sets X, Y holds $\mathrm{card}[\!:X, Y:\!] = \mathrm{card}\,X \cdot \mathrm{card}\,Y$.

(67)[7] For all finite sets X, Y such that $X \subset Y$ holds $\mathrm{card}\,X < \mathrm{card}\,Y$ and $\overline{\overline{X}} < \overline{\overline{Y}}$.

(68) If $\overline{\overline{X}} \leq \overline{\overline{Y}}$ or $\overline{\overline{X}} < \overline{\overline{Y}}$ and if Y is finite, then X is finite.

In the sequel x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 denote sets.
Next we state a number of propositions:

(69) $\mathrm{card}\{x_1, x_2\} \leq 2$.

(70) $\mathrm{card}\{x_1, x_2, x_3\} \leq 3$.

(71) $\mathrm{card}\{x_1, x_2, x_3, x_4\} \leq 4$.

(72) $\mathrm{card}\{x_1, x_2, x_3, x_4, x_5\} \leq 5$.

[6] The propositions (55) and (56) have been removed.
[7] The proposition (66) has been removed.

(73) $\text{card}\{x_1,x_2,x_3,x_4,x_5,x_6\} \leq 6$.

(74) $\text{card}\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\} \leq 7$.

(75) $\text{card}\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\} \leq 8$.

(76) If $x_1 \neq x_2$, then $\text{card}\{x_1,x_2\} = 2$.

(77) If $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$, then $\text{card}\{x_1,x_2,x_3\} = 3$.

(78) If $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$, then $\text{card}\{x_1,x_2,x_3,x_4\} = 4$.

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