JOURNAL OF FORMALIZED MATHEMATICS
Volume 1, Released 1989, Published 2003
Inst. of Computer Science, Univ. of Białystok

Cardinal Numbers

Grzegorz Bancerek
Warsaw University
Białystok

Summary. We present the choice function rule in the beginning of the article. In the main part of the article we formalize the base of cardinal theory. In the first section we introduce the concept of cardinal numbers and order relations between them. We present here Cantor-Bernstein theorem and other properties of order relation of cardinals. In the second section we show that every set has cardinal number equipotence to it. We introduce notion of alephs and we deal with the concept of finite set. At the end of the article we show two schemes of cardinal induction. Some definitions are based on [9] and [10].

MML Identifier: `CARD_1`.

WWW: `http://mizar.org/JFM/Vol1/card_1.html`

The articles [11], [7], [13], [12], [1], [14], [6], [4], [2], [3], [5], and [8] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: A, B denote ordinal numbers, X, X_1, Y, Y_1, Z, x, y denote sets, R denotes a binary relation, f denotes a function, and m, n denote natural numbers.

Let I_1 be a set. We say that I_1 is cardinal if and only if:

(Def. 1) There exists B such that $I_1 = B$ and for every A such that $A \approx B$ holds $B \subseteq A$.

Let us observe that there exists a set which is cardinal.

A cardinal number is a cardinal set.

Let us observe that every set which is cardinal is also ordinal.

In the sequel M, N denote cardinal numbers.

We now state the proposition

(4)[1] For every X there exists A such that $X \approx A$.

Let us consider M, N. We introduce $M \leq N$ as a synonym of $M \subseteq N$. We introduce $M < N$ as a synonym of $M \in N$.

We now state three propositions:

(8)[2] $M = N$ iff $M \approx N$.

(13)[3] $M < N$ iff $M \leq N$ and $M \neq N$.

(14) $M < N$ iff $N \nleq M$.

Let us consider X. The functor $\overline{\overline{X}}$ yields a cardinal number and is defined as follows:

[1] The propositions (1)–(3) have been removed.
[2] The propositions (5)–(7) have been removed.
[3] The propositions (9)–(12) have been removed.

(Def. 5)[4] $X \approx \overline{\overline{X}}$.

Next we state a number of propositions:

(21)[5] $X \approx Y$ iff $\overline{\overline{X}} = \overline{\overline{Y}}$.

(22) If R is well-ordering, then field $R \approx \overline{R}$.

(23) If $X \subseteq M$, then $\overline{\overline{X}} \leq M$.

(24) $\overline{\overline{A}} \subseteq A$.

(25) If $X \in M$, then $\overline{\overline{X}} < M$.

(26) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff there exists f such that f is one-to-one and $\mathrm{dom}\, f = X$ and $\mathrm{rng}\, f \subseteq Y$.

(27) If $X \subseteq Y$, then $\overline{\overline{X}} \leq \overline{\overline{Y}}$.

(28) $\overline{\overline{X}} \leq \overline{\overline{Y}}$ iff there exists f such that $\mathrm{dom}\, f = Y$ and $X \subseteq \mathrm{rng}\, f$.

(29) $X \not\approx 2^X$.

(30) $\overline{\overline{X}} < \overline{\overline{2^X}}$.

Let us consider X. The functor X^+ yields a cardinal number and is defined by:

(Def. 6) $\overline{\overline{X}} < X^+$ and for every M such that $\overline{\overline{X}} < M$ holds $X^+ \leq M$.

One can prove the following propositions:

(32)[6] $M < M^+$.

(33) $\overline{\overline{\emptyset}} < X^+$.

(34) If $\overline{\overline{X}} = \overline{\overline{Y}}$, then $X^+ = Y^+$.

(35) If $X \approx Y$, then $X^+ = Y^+$.

(36) $A \in A^+$.

In the sequel L denotes a transfinite sequence.
Let us consider M. We say that M is limit if and only if:

(Def. 7) It is not true that there exists N such that $M = N^+$.

We introduce M is a limit cardinal number as a synonym of M is limit.
Let us consider A. The functor $\aleph_A$ yielding a set is defined by the condition (Def. 8).

(Def. 8) There exists L such that

(i) $\aleph_A = \mathrm{last}\, L$,

(ii) $\mathrm{dom}\, L = \mathrm{succ}\, A$,

(iii) $L(\emptyset) = \overline{\overline{\mathbb{N}}}$,

(iv) for every B such that $\mathrm{succ}\, B \in \mathrm{succ}\, A$ holds $L(\mathrm{succ}\, B) = (\bigcup\{L(B)\})^+$, and

(v) for every B such that $B \in \mathrm{succ}\, A$ and $B \neq \emptyset$ and B is a limit ordinal number holds $L(B) = \overline{\overline{\sup(L \restriction B)}}$.

[4] The definitions (Def. 2)–(Def. 4) have been removed.
[5] The propositions (15)–(20) have been removed.
[6] The proposition (31) has been removed.

Let us consider A. Note that $\aleph_A$ is cardinal.

One can prove the following propositions:

(38)[7] $\aleph_0 = \overline{\overline{\mathbb{N}}}$.

(39) $\aleph_{\mathrm{succ}\, A} = (\aleph_A)^+$.

(40) Suppose $A \neq \emptyset$ and A is a limit ordinal number. Let given L. If $\mathrm{dom}\, L = A$ and for every B such that $B \in A$ holds $L(B) = \aleph_B$, then $\aleph_A = \overline{\overline{\sup L}}$.

(41) $A \in B$ iff $\aleph_A < \aleph_B$.

(42) If $\aleph_A = \aleph_B$, then $A = B$.

(43) $A \subseteq B$ iff $\aleph_A \leq \aleph_B$.

(44) If $X \subseteq Y$ and $Y \subseteq Z$ and $X \approx Z$, then $X \approx Y$ and $Y \approx Z$.

(45) If $2^Y \subseteq X$, then $\overline{\overline{Y}} < \overline{\overline{X}}$ and $Y \not\approx X$.

(46) $X \approx \emptyset$ iff $X = \emptyset$.

(47) $\overline{\overline{\emptyset}} = \emptyset$.

(48) $X \approx \{x\}$ iff there exists x such that $X = \{x\}$.

(49) $\overline{\overline{X}} = \overline{\overline{\{x\}}}$ iff there exists x such that $X = \{x\}$.

(50) $\overline{\overline{\{x\}}} = \mathbf{1}$.

(51) $0 = \emptyset$.

(52) $\mathrm{succ}\, n = n + 1$.

(54)[8] If A is natural, then there exists n such that $n = A$.

(56)[9] $n \leq m$ iff $n \subseteq m$.

(58)[10] If X misses X_1 and Y misses Y_1 and $X \approx Y$ and $X_1 \approx Y_1$, then $X \cup X_1 \approx Y \cup Y_1$.

(59) If $x \in X$ and $y \in X$, then $X \setminus \{x\} \approx X \setminus \{y\}$.

(60) If $X \subseteq \mathrm{dom}\, f$ and f is one-to-one, then $X \approx f^\circ X$.

(61) If $X \approx Y$ and $x \in X$ and $y \in Y$, then $X \setminus \{x\} \approx Y \setminus \{y\}$.

(64)[11] If $n \approx m$, then $n = m$.

(65) If $x \in \omega$, then x is cardinal.

One can verify that every natural number is cardinal.

Next we state several propositions:

(66) For every natural number n holds $n = \overline{\overline{n}}$.

(68)[12] If $X \approx Y$ and X is finite, then Y is finite.

(69) n is finite and $\overline{\overline{n}}$ is finite.

[7] The proposition (37) has been removed.
[8] The proposition (53) has been removed.
[9] The proposition (55) has been removed.
[10] The proposition (57) has been removed.
[11] The propositions (62) and (63) have been removed.
[12] The proposition (67) has been removed.

(71)[13] If $\overline{\overline{n}} = \overline{\overline{m}}$, then $n = m$.

(72) $\overline{\overline{n}} \leq \overline{\overline{m}}$ iff $n \leq m$.

(73) $\overline{\overline{n}} < \overline{\overline{m}}$ iff $n < m$.

(74) If X is finite, then there exists n such that $X \approx n$.

(76)[14] $\overline{\overline{n}}^{+} = \overline{\overline{n+1}}$.

Let X be a finite set. Then $\overline{\overline{X}}$ is a natural number and it can be characterized by the condition:

(Def. 11)[15] $\overline{\overline{\overline{\overline{X}}}} = \overline{\overline{X}}$.

We introduce $\operatorname{card} X$ as a synonym of $\overline{\overline{X}}$.

We now state several propositions:

(78)[16] $\operatorname{card} \emptyset = 0$.

(79) $\operatorname{card}\{x\} = 1$.

(80) For all finite sets X, Y such that $X \subseteq Y$ holds $\operatorname{card} X \leq \operatorname{card} Y$.

(81) For all finite sets X, Y such that $X \approx Y$ holds $\operatorname{card} X = \operatorname{card} Y$.

(82) If X is finite, then X^{+} is finite.

In this article we present several logical schemes. The scheme *Cardinal Ind* concerns a unary predicate $\mathcal{P}$, and states that:

For every M holds $\mathcal{P}[M]$

provided the parameters meet the following conditions:

- $\mathcal{P}[\emptyset]$,
- For every M such that $\mathcal{P}[M]$ holds $\mathcal{P}[M^{+}]$, and
- For every M such that $M \neq \emptyset$ and M is a limit cardinal number and for every N such that $N < M$ holds $\mathcal{P}[N]$ holds $\mathcal{P}[M]$.

The scheme *Cardinal CompInd* concerns a unary predicate $\mathcal{P}$, and states that:

For every M holds $\mathcal{P}[M]$

provided the parameters satisfy the following condition:

- For every M such that for every N such that $N < M$ holds $\mathcal{P}[N]$ holds $\mathcal{P}[M]$.

Next we state three propositions:

(83) $\aleph_0 = \omega$.

(84) $\overline{\overline{\omega}} = \omega$.

(85) $\overline{\overline{\omega}}$ is a limit cardinal number.

Let us note that every natural number is finite.

Let us observe that there exists a cardinal number which is finite.

Next we state the proposition

(86) For every finite cardinal number M there exists n such that $M = \overline{\overline{n}}$.

Let X be a finite set. Note that $\overline{\overline{X}}$ is finite.

[13] The proposition (70) has been removed.
[14] The proposition (75) has been removed.
[15] The definitions (Def. 9) and (Def. 10) have been removed.
[16] The proposition (77) has been removed.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[3] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[4] Grzegorz Bancerek. The well ordering relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/wellord1.html`.

[5] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/wellord2.html`.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[9] Wojciech Guzicki and Paweł Zbierski. *Podstawy teorii mnogości*. PWN, Warszawa, 1978.

[10] Kazimierz Kuratowski and Andrzej Mostowski. *Teoria mnogości*. PTM, Wrocław, 1952.

[11] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[12] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[14] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received September 19, 1989

Published January 2, 2004