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A Theory of Boolean Valued Functions and Partitions

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Summary. In this paper, we define Boolean valued functions. Some of their algebraic properties are proved. We also introduce and examine the infimum and supremum of Boolean valued functions and their properties. In the last section, relations between Boolean valued functions and partitions are discussed.

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The articles [11], [4], [13], [1], [16], [15], [14], [2], [3], [9], [12], [8], [10], [7], [5], and [6] provide the notation and terminology for this paper.

1. Boolean Operations

In this paper Y denotes a set.

Let k, l be boolean sets. The functor $k \Rightarrow l$ is defined as follows:

(Def. 1) $k \Rightarrow l = \neg k \vee l$.

The functor $k \Leftrightarrow l$ is defined as follows:

(Def. 2) $k \Leftrightarrow l = \neg(k \oplus l)$.

Let us note that the functor $k \Leftrightarrow l$ is commutative.

Let k, l be boolean sets. Note that $k \Rightarrow l$ is boolean and $k \Leftrightarrow l$ is boolean.

Let us note that every set which is boolean is also natural.

Let k, l be boolean sets. Let us observe that $k \leq l$ if and only if:

(Def. 3) $k \Rightarrow l = true$.

We introduce $k \Subset l$ as a synonym of $k \leq l$.

2. Boolean Valued Functions

Let us consider Y. The functor $\mathrm{BVF}(Y)$ is defined as follows:

(Def. 4) $\mathrm{BVF}(Y) = Boolean^Y$.

Let Y be a set. Note that $\mathrm{BVF}(Y)$ is functional and non empty.

Let Y be a set. Note that every element of $\mathrm{BVF}(Y)$ is boolean-valued.

In the sequel Y is a non empty set.

Let a be a boolean-valued function and let x be a set. We introduce $\mathrm{Pj}(a,x)$ as a synonym of $a(x)$.

Let us consider Y and let a be an element of $\mathrm{BVF}(Y)$. Then $\neg a$ is an element of $\mathrm{BVF}(Y)$. Let b be an element of $\mathrm{BVF}(Y)$. Then $a \wedge b$ is an element of $\mathrm{BVF}(Y)$.

Let p, q be boolean-valued functions. The functor $p \vee q$ yielding a function is defined as follows:

(Def. 5) $\mathrm{dom}(p \vee q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \vee q)$ holds $(p \vee q)(x) = p(x) \vee q(x)$.

Let us note that the functor $p \vee q$ is commutative. The functor $p \oplus q$ yields a function and is defined by:

(Def. 6) $\mathrm{dom}(p \oplus q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \oplus q)$ holds $(p \oplus q)(x) = p(x) \oplus q(x)$.

Let us note that the functor $p \oplus q$ is commutative.

Let p, q be boolean-valued functions. One can check that $p \vee q$ is boolean-valued and $p \oplus q$ is boolean-valued.

Let A be a non empty set and let p, q be elements of $Boolean^A$. Then $p \vee q$ is an element of $Boolean^A$ and it can be characterized by the condition:

(Def. 7) For every element x of A holds $(p \vee q)(x) = p(x) \vee q(x)$.

Then $p \oplus q$ is an element of $Boolean^A$ and it can be characterized by the condition:

(Def. 8) For every element x of A holds $(p \oplus q)(x) = p(x) \oplus q(x)$.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. Then $a \vee b$ is an element of $\mathrm{BVF}(Y)$. Then $a \oplus b$ is an element of $\mathrm{BVF}(Y)$.

Let p, q be boolean-valued functions. The functor $p \Rightarrow q$ yields a function and is defined as follows:

(Def. 9) $\mathrm{dom}(p \Rightarrow q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \Rightarrow q)$ holds $(p \Rightarrow q)(x) = p(x) \Rightarrow q(x)$.

The functor $p \Leftrightarrow q$ yielding a function is defined as follows:

(Def. 10) $\mathrm{dom}(p \Leftrightarrow q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \Leftrightarrow q)$ holds $(p \Leftrightarrow q)(x) = p(x) \Leftrightarrow q(x)$.

Let us note that the functor $p \Leftrightarrow q$ is commutative.

Let p, q be boolean-valued functions. One can check that $p \Rightarrow q$ is boolean-valued and $p \Leftrightarrow q$ is boolean-valued.

Let A be a non empty set and let p, q be elements of $Boolean^A$. Then $p \Rightarrow q$ is an element of $Boolean^A$ and it can be characterized by the condition:

(Def. 11) For every element x of A holds $(p \Rightarrow q)(x) = \neg \mathrm{Pj}(p,x) \vee \mathrm{Pj}(q,x)$.

Then $p \Leftrightarrow q$ is an element of $Boolean^A$ and it can be characterized by the condition:

(Def. 12) For every element x of A holds $(p \Leftrightarrow q)(x) = \neg(\mathrm{Pj}(p,x) \oplus \mathrm{Pj}(q,x))$.

Let us consider Y and let a, b be elements of $\mathrm{BVF}(Y)$. Then $a \Rightarrow b$ is an element of $\mathrm{BVF}(Y)$. Then $a \Leftrightarrow b$ is an element of $\mathrm{BVF}(Y)$.

Let us consider Y. The functor $false(Y)$ yields an element of $Boolean^Y$ and is defined by:

(Def. 13) For every element x of Y holds $\mathrm{Pj}(false(Y),x) = false$.

Let us consider Y. The functor $true(Y)$ yielding an element of $Boolean^Y$ is defined as follows:

(Def. 14) For every element x of Y holds $\mathrm{Pj}(true(Y),x) = true$.

One can prove the following propositions:

(4)[1] For every boolean-valued function a holds $\neg\neg a = a$.

[1] The propositions (1)–(3) have been removed.

(5) For every element a of $Boolean^Y$ holds $\neg true(Y) = false(Y)$ and $\neg false(Y) = true(Y)$.

(6) For all elements a, b of $Boolean^Y$ holds $a \wedge a = a$.

(7) For all elements a, b, c of $Boolean^Y$ holds $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

(8) For every element a of $Boolean^Y$ holds $a \wedge false(Y) = false(Y)$.

(9) For every element a of $Boolean^Y$ holds $a \wedge true(Y) = a$.

(10) For every element a of $Boolean^Y$ holds $a \vee a = a$.

(11) For all elements a, b, c of $Boolean^Y$ holds $(a \vee b) \vee c = a \vee (b \vee c)$.

(12) For every element a of $Boolean^Y$ holds $a \vee false(Y) = a$.

(13) For every element a of $Boolean^Y$ holds $a \vee true(Y) = true(Y)$.

(14) For all elements a, b, c of $Boolean^Y$ holds $a \wedge b \vee c = (a \vee c) \wedge (b \vee c)$.

(15) For all elements a, b, c of $Boolean^Y$ holds $(a \vee b) \wedge c = a \wedge c \vee b \wedge c$.

(16) For all elements a, b of $Boolean^Y$ holds $\neg(a \vee b) = \neg a \wedge \neg b$.

(17) For all elements a, b of $Boolean^Y$ holds $\neg(a \wedge b) = \neg a \vee \neg b$.

Let us consider Y and let a, b be elements of $Boolean^Y$. The predicate $a \Subset b$ is defined as follows:

(Def. 15) For every element x of Y such that $\mathrm{Pj}(a,x) = true$ holds $\mathrm{Pj}(b,x) = true$.

Let us note that the predicate $a \Subset b$ is reflexive.

The following propositions are true:

(18) For all elements a, b, c of $Boolean^Y$ holds if $a \Subset b$ and $b \Subset a$, then $a = b$ and if $a \Subset b$ and $b \Subset c$, then $a \Subset c$.

(19) For all elements a, b of $Boolean^Y$ holds $a \Rightarrow b = true(Y)$ iff $a \Subset b$.

(20) For all elements a, b of $Boolean^Y$ holds $a \Leftrightarrow b = true(Y)$ iff $a = b$.

(21) For every element a of $Boolean^Y$ holds $false(Y) \Subset a$ and $a \Subset true(Y)$.

3. Infimum and Supremum

Let us consider Y and let a be an element of $Boolean^Y$. The functor $\mathrm{INF}\,a$ yields an element of $Boolean^Y$ and is defined as follows:

(Def. 16) $$\mathrm{INF}\,a = \begin{cases} true(Y), & \text{if foreveryelement } x \text{ of } Y \text{ holds } \mathrm{Pj}(a,x) = true, \\ false(Y), & \text{otherwise.} \end{cases}$$

The functor $\mathrm{SUP}\,a$ yields an element of $Boolean^Y$ and is defined by:

(Def. 17) $$\mathrm{SUP}\,a = \begin{cases} false(Y), & \text{if foreveryelement } x \text{ of } Y \text{ holds } \mathrm{Pj}(a,x) = false, \\ true(Y), & \text{otherwise.} \end{cases}$$

We now state two propositions:

(22) For every element a of $Boolean^Y$ holds $\neg\mathrm{INF}\,a = \mathrm{SUP}\,\neg a$ and $\neg\mathrm{SUP}\,a = \mathrm{INF}\,\neg a$.

(23) $\mathrm{INF}\,false(Y) = false(Y)$ and $\mathrm{INF}\,true(Y) = true(Y)$ and $\mathrm{SUP}\,false(Y) = false(Y)$ and $\mathrm{SUP}\,true(Y) = true(Y)$.

Let us consider Y. Observe that $false(Y)$ is constant.
Let us consider Y. One can check that $true(Y)$ is constant.
Let Y be a non empty set. Note that there exists an element of $Boolean^Y$ which is constant.
We now state several propositions:

(24) For every constant element a of $Boolean^Y$ holds $a = false(Y)$ or $a = true(Y)$.

(25) For every constant element d of $Boolean^Y$ holds $\mathrm{INF}\,d = d$ and $\mathrm{SUP}\,d = d$.

(26) For all elements a, b of $Boolean^Y$ holds $\mathrm{INF}(a \wedge b) = \mathrm{INF}\,a \wedge \mathrm{INF}\,b$ and $\mathrm{SUP}(a \vee b) = \mathrm{SUP}\,a \vee \mathrm{SUP}\,b$.

(27) For every element a of $Boolean^Y$ and for every constant element d of $Boolean^Y$ holds $\mathrm{INF}(d \Rightarrow a) = d \Rightarrow \mathrm{INF}\,a$ and $\mathrm{INF}(a \Rightarrow d) = \mathrm{SUP}\,a \Rightarrow d$.

(28) For every element a of $Boolean^Y$ and for every constant element d of $Boolean^Y$ holds $\mathrm{INF}(d \vee a) = d \vee \mathrm{INF}\,a$ and $\mathrm{SUP}(d \wedge a) = d \wedge \mathrm{SUP}\,a$ and $\mathrm{SUP}(a \wedge d) = \mathrm{SUP}\,a \wedge d$.

(29) For every element a of $Boolean^Y$ and for every element x of Y holds $\mathrm{Pj}(\mathrm{INF}\,a, x) \Subset \mathrm{Pj}(a, x)$.

(30) For every element a of $Boolean^Y$ and for every element x of Y holds $\mathrm{Pj}(a, x) \Subset \mathrm{Pj}(\mathrm{SUP}\,a, x)$.

4. Boolean Valued Functions and Partitions

Let us consider Y, let a be an element of $Boolean^Y$, and let P_1 be a partition of Y. We say that a is dependent of P_1 if and only if:

(Def. 18) For every set F such that $F \in P_1$ and for all sets x_1, x_2 such that $x_1 \in F$ and $x_2 \in F$ holds $a(x_1) = a(x_2)$.

Next we state two propositions:

(31) For every element a of $Boolean^Y$ holds a is dependent of $I(Y)$.

(32) For every constant element a of $Boolean^Y$ holds a is dependent of $O(Y)$.

Let us consider Y and let P_1 be a partition of Y. We see that the element of P_1 is a subset of Y.
Let us consider Y, let x be an element of Y, and let P_1 be a partition of Y. Then $\mathrm{EqClass}(x, P_1)$ is an element of P_1. We introduce $\mathrm{Lift}(x, P_1)$ as a synonym of $\mathrm{EqClass}(x, P_1)$.
Let us consider Y, let a be an element of $Boolean^Y$, and let P_1 be a partition of Y. The functor $\mathrm{INF}(a, P_1)$ yields an element of $Boolean^Y$ and is defined by the condition (Def. 19).

(Def. 19) Let y be an element of Y. Then

(i) if for every element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ holds $\mathrm{Pj}(a, x) = true$, then $\mathrm{Pj}(\mathrm{INF}(a, P_1), y) = true$, and

(ii) if it is not true that for every element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ holds $\mathrm{Pj}(a, x) = true$, then $\mathrm{Pj}(\mathrm{INF}(a, P_1), y) = false$.

Let us consider Y, let a be an element of $Boolean^Y$, and let P_1 be a partition of Y. The functor $\mathrm{SUP}(a, P_1)$ yields an element of $Boolean^Y$ and is defined by the condition (Def. 20).

(Def. 20) Let y be an element of Y. Then

(i) if there exists an element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ and $\mathrm{Pj}(a, x) = true$, then $\mathrm{Pj}(\mathrm{SUP}(a, P_1), y) = true$, and

(ii) if it is not true that there exists an element x of Y such that $x \in \mathrm{EqClass}(y, P_1)$ and $\mathrm{Pj}(a, x) = true$, then $\mathrm{Pj}(\mathrm{SUP}(a, P_1), y) = false$.

Next we state a number of propositions:

(33) For every element a of $Boolean^Y$ and for every partition P_1 of Y holds $\mathrm{INF}(a,P_1)$ is dependent of P_1.

(34) For every element a of $Boolean^Y$ and for every partition P_1 of Y holds $\mathrm{SUP}(a,P_1)$ is dependent of P_1.

(35) For every element a of $Boolean^Y$ and for every partition P_1 of Y holds $\mathrm{INF}(a,P_1) \Subset a$.

(36) For every element a of $Boolean^Y$ and for every partition P_1 of Y holds $a \Subset \mathrm{SUP}(a,P_1)$.

(37) For every element a of $Boolean^Y$ and for every partition P_1 of Y holds $\neg \mathrm{INF}(a,P_1) = \mathrm{SUP}(\neg a,P_1)$.

(38) For every element a of $Boolean^Y$ holds $\mathrm{INF}(a, O(Y)) = \mathrm{INF}\, a$.

(39) For every element a of $Boolean^Y$ holds $\mathrm{SUP}(a, O(Y)) = \mathrm{SUP}\, a$.

(40) For every element a of $Boolean^Y$ holds $\mathrm{INF}(a, I(Y)) = a$.

(41) For every element a of $Boolean^Y$ holds $\mathrm{SUP}(a, I(Y)) = a$.

(42) For all elements a, b of $Boolean^Y$ and for every partition P_1 of Y holds $\mathrm{INF}(a \wedge b,P_1) = \mathrm{INF}(a,P_1) \wedge \mathrm{INF}(b,P_1)$.

(43) For all elements a, b of $Boolean^Y$ and for every partition P_1 of Y holds $\mathrm{SUP}(a \vee b,P_1) = \mathrm{SUP}(a,P_1) \vee \mathrm{SUP}(b,P_1)$.

Let us consider Y and let f be an element of $Boolean^Y$. The functor $\mathrm{GPart}\, f$ yielding a partition of Y is defined by:

(Def. 21) $\mathrm{GPart}\, f = \{\{x; x$ ranges over elements of Y: $f(x) = true\}, \{x'; x'$ ranges over elements of Y: $f(x') = false\}\} \setminus \{\emptyset\}$.

The following two propositions are true:

(44) For every element a of $Boolean^Y$ holds a is dependent of $\mathrm{GPart}\, a$.

(45) For every element a of $Boolean^Y$ and for every partition P_1 of Y such that a is dependent of P_1 holds P_1 is finer than $\mathrm{GPart}\, a$.

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