JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
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On the Subcontinua of a Real Line[1]

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Summary. In [11] we showed that the only proper subcontinua of the simple closed curve are arcs and single points. In this article we prove that the only proper subcontinua of the real line are closed intervals. We introduce some auxiliary notions such as $]a,b[_{\mathbb{Q}}$, $]a,b[_{\mathbb{IQ}}$ – intervals consisting of rational and irrational numbers respectively. We show also some basic topological properties of intervals.

MML Identifier: BORSUK_5.

WWW: http://mizar.org/JFM/Vol15/borsuk_5.html

The articles [23], [27], [2], [24], [22], [25], [28], [4], [5], [26], [19], [7], [21], [14], [17], [18], [1], [9], [6], [10], [15], [8], [20], [16], [13], [12], and [3] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(2)[1] For all sets A, B, a such that $A \subseteq B$ and $B \subseteq A \cup \{a\}$ holds $A \cup \{a\} = B$ or $A = B$.

(3) For all sets x_1, x_2, x_3, x_4, x_5, x_6 holds $\{x_1,x_2,x_3,x_4,x_5,x_6\} = \{x_1,x_3,x_6\} \cup \{x_2,x_4,x_5\}$.

In the sequel x_1, x_2, x_3, x_4, x_5, x_6, x_7 are sets.

Let x_1, x_2, x_3, x_4, x_5, x_6 be sets. We say that x_1, x_2, x_3, x_4, x_5, x_6 are mutually different if and only if the conditions (Def. 1) are satisfied.

(Def. 1) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$ and $x_4 \neq x_5$ and $x_4 \neq x_6$ and $x_5 \neq x_6$.

Let x_1, x_2, x_3, x_4, x_5, x_6, x_7 be sets. We say that x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different if and only if the conditions (Def. 2) are satisfied.

(Def. 2) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_1 \neq x_7$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_2 \neq x_7$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$ and $x_3 \neq x_7$ and $x_4 \neq x_5$ and $x_4 \neq x_6$ and $x_4 \neq x_7$ and $x_5 \neq x_6$ and $x_5 \neq x_7$ and $x_6 \neq x_7$.

The following propositions are true:

(4) For all sets x_1, x_2, x_3, x_4, x_5, x_6 such that x_1, x_2, x_3, x_4, x_5, x_6 are mutually different holds $\mathrm{card}\{x_1,x_2,x_3,x_4,x_5,x_6\} = 6$.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.
[1] The proposition (1) has been removed.

(5) For all sets x_1, x_2, x_3, x_4, x_5, x_6, x_7 such that x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different holds $\mathrm{card}\{x_1,x_2,x_3,x_4,x_5,x_6,x_7\} = 7$.

(6) If $\{x_1,x_2,x_3\}$ misses $\{x_4,x_5,x_6\}$, then $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_1 \neq x_6$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_2 \neq x_6$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_3 \neq x_6$.

(7) Suppose x_1, x_2, x_3 are mutually different and x_4, x_5, x_6 are mutually different and $\{x_1,x_2,x_3\}$ misses $\{x_4,x_5,x_6\}$. Then x_1, x_2, x_3, x_4, x_5, x_6 are mutually different.

(8) Suppose x_1, x_2, x_3, x_4, x_5, x_6 are mutually different and $\{x_1,x_2,x_3,x_4,x_5,x_6\}$ misses $\{x_7\}$. Then x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different.

(9) If x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different, then x_7, x_1, x_2, x_3, x_4, x_5, x_6 are mutually different.

(10) If x_1, x_2, x_3, x_4, x_5, x_6, x_7 are mutually different, then x_1, x_2, x_5, x_3, x_6, x_7, x_4 are mutually different.

(11) Let T be a non empty topological space and a, b be points of T. Given a map f from $\mathbb{I}$ into T such that f is continuous and $f(0) = a$ and $f(1) = b$. Then there exists a map g from $\mathbb{I}$ into T such that g is continuous and $g(0) = b$ and $g(1) = a$.

Let us observe that $\mathbb{R}^{\mathbf{1}}$ is arcwise connected.
Let us observe that there exists a topological space which is connected and non empty.

2. Intervals

We now state two propositions:

(12) Every subset of $\mathbb{R}$ is a subset of $\mathbb{R}^{\mathbf{1}}$.

(13) $\Omega_{\mathbb{R}^{\mathbf{1}}} = \mathbb{R}$.

Let a be a real number. We introduce $]-\infty,a]$ as a synonym of $]-\infty,a]$. We introduce $]-\infty,a[$ as a synonym of $]-\infty,a[$. We introduce $[a,+\infty[$ as a synonym of $[a,+\infty[$. We introduce $]a,+\infty[$ as a synonym of $]a,+\infty[$.

One can prove the following propositions:

(14) For all real numbers a, b holds $a \in]b,+\infty[$ iff $a > b$.

(15) For all real numbers a, b holds $a \in [b,+\infty[$ iff $a \geq b$.

(16) For all real numbers a, b holds $a \in]-\infty,b]$ iff $a \leq b$.

(17) For all real numbers a, b holds $a \in]-\infty,b[$ iff $a < b$.

(18) For every real number a holds $\mathbb{R} \setminus \{a\} =]-\infty,a[\cup]a,+\infty[$.

(19) For all real numbers a, b, c, d such that $a < b$ and $b \leq c$ holds $[a,b]$ misses $]c,d]$.

(20) For all real numbers a, b, c, d such that $a < b$ and $b \leq c$ holds $[a,b[$ misses $[c,d]$.

(21) Let A, B be subsets of $\mathbb{R}^{\mathbf{1}}$ and a, b, c, d be real numbers. Suppose $a < b$ and $b \leq c$ and $c < d$ and $A = [a,b[$ and $B =]c,d]$. Then A and B are separated.

(22) For every real number a holds $\mathbb{R} \setminus]-\infty,a[= [a,+\infty[$.

(23) For every real number a holds $\mathbb{R} \setminus]-\infty,a] =]a,+\infty[$.

(24) For every real number a holds $\mathbb{R} \setminus]a,+\infty[=]-\infty,a]$.

(25) For every real number a holds $\mathbb{R} \setminus [a,+\infty[=]-\infty,a[$.

(26) For every real number a holds $]-\infty, a]$ misses $]a, +\infty[$.

(27) For every real number a holds $]-\infty, a[$ misses $[a, +\infty[$.

(28) For all real numbers a, b, c such that $a \leq c$ and $c \leq b$ holds $[a,b] \cup [c, +\infty[= [a, +\infty[$.

(29) For all real numbers a, b, c such that $a \leq c$ and $c \leq b$ holds $]-\infty, c] \cup [a,b] =]-\infty, b]$.

(30) For every 1-sorted structure T and for every subset A of T holds $\{A\}$ is a family of subsets of T.

(31) For every 1-sorted structure T and for all subsets A, B of T holds $\{A, B\}$ is a family of subsets of T.

(32) For every 1-sorted structure T and for all subsets A, B, C of T holds $\{A, B, C\}$ is a family of subsets of T.

Let us observe that every element of $\mathbb{Q}$ is real.
Let us note that every element of the metric space of real numbers is real.
The following propositions are true:

(33) Let A be a subset of $\mathbb{R}^{\mathbf{1}}$ and p be a point of the metric space of real numbers. Then $p \in \overline{A}$ if and only if for every real number r such that $r > 0$ holds $\mathrm{Ball}(p, r)$ meets A.

(34) For all elements p, q of the metric space of real numbers such that $q \geq p$ holds $\rho(p,q) = q - p$.

(35) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{Q}$ holds $\overline{A} =$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(36) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A =]a, b[$ and $a \neq b$ holds $\overline{A} = [a, b]$.

3. Rational and Irrational Numbers

Let us observe that e is irrational.
The subset $\mathbb{IQ}$ of $\mathbb{R}$ is defined as follows:

(Def. 3) $\mathbb{IQ} = \mathbb{R} \setminus \mathbb{Q}$.

Let a, b be real numbers. The functor $]a, b[_{\mathbb{Q}}$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 4) $]a, b[_{\mathbb{Q}} = \mathbb{Q} \cap]a, b[$.

The functor $]a, b[_{\mathbb{IQ}}$ yielding a subset of $\mathbb{R}$ is defined by:

(Def. 5) $]a, b[_{\mathbb{IQ}} = \mathbb{IQ} \cap]a, b[$.

One can prove the following proposition

(37) For every real number x holds x is irrational iff $x \in \mathbb{IQ}$.

Let us observe that there exists a real number which is irrational.
One can verify that $\mathbb{IQ}$ is non empty.
Next we state several propositions:

(38) For every rational number a and for every irrational real number b holds $a + b$ is irrational.

(39) For every irrational real number a holds $-a$ is irrational.

(40) For every rational number a and for every irrational real number b holds $a - b$ is irrational.

(41) For every rational number a and for every irrational real number b holds $b - a$ is irrational.

(42) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $a \cdot b$ is irrational.

(43) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $\frac{b}{a}$ is irrational.

One can verify that every real number which is irrational is also non zero.
Next we state two propositions:

(44) For every rational number a and for every irrational real number b such that $a \neq 0$ holds $\frac{a}{b}$ is irrational.

(45) For every irrational real number r holds $\operatorname{frac} r$ is irrational.

Let r be an irrational real number. Note that $\operatorname{frac} r$ is irrational.
Let a be an irrational real number. One can check that $-a$ is irrational.
Let a be a rational number and let b be an irrational real number. One can verify the following observations:

* $a+b$ is irrational,
* $b+a$ is irrational,
* $a-b$ is irrational, and
* $b-a$ is irrational.

Let us note that there exists a rational number which is non zero.
Let a be a non zero rational number and let b be an irrational real number. One can verify the following observations:

* $a \cdot b$ is irrational,
* $b \cdot a$ is irrational,
* $\frac{a}{b}$ is irrational, and
* $\frac{b}{a}$ is irrational.

The following propositions are true:

(46) For every irrational real number r holds $0 < \operatorname{frac} r$.

(47) For all real numbers a, b such that $a < b$ there exist rational numbers p_1, p_2 such that $a < p_1$ and $p_1 < p_2$ and $p_2 < b$.

(48) For all real numbers s_1, s_3, s_4, l such that $s_1 \leq s_3$ and $s_1 < s_4$ and $0 < l$ and $l < 1$ holds $s_1 < (1-l) \cdot s_3 + l \cdot s_4$.

(49) For all real numbers s_1, s_3, s_4, l such that $s_3 < s_1$ and $s_4 \leq s_1$ and $0 < l$ and $l < 1$ holds $(1-l) \cdot s_3 + l \cdot s_4 < s_1$.

(50) For all real numbers a, b such that $a < b$ there exists an irrational real number p such that $a < p$ and $p < b$.

(51) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = \mathbb{IQ}$ holds $\overline{A} =$ the carrier of $\mathbb{R}^{\mathbf{1}}$.

(52) For all real numbers a, b, c such that $a < b$ holds $c \in]a,b[_{\mathbb{Q}}$ iff c is rational and $a < c$ and $c < b$.

(53) For all real numbers a, b, c such that $a < b$ holds $c \in]a,b[_{\mathbb{IQ}}$ iff c is irrational and $a < c$ and $c < b$.

(54) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a,b[_{\mathbb{Q}}$ holds $\overline{A} = [a,b]$.

(55) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a < b$ and $A =]a,b[_{\mathbb{IQ}}$ holds $\overline{A} = [a,b]$.

(56) For every connected topological space T and for every closed open subset A of T holds $A = \emptyset$ or $A = \Omega_T$.

(57) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that A is closed and open holds $A = \emptyset$ or $A = \mathbb{R}$.

4. Topological Properties of Intervals

Next we state a number of propositions:

(58) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A = [a,b[$ and $a \neq b$ holds $\overline{A} = [a,b]$.

(59) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $A =]a,b]$ and $a \neq b$ holds $\overline{A} = [a,b]$.

(60) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b, c such that $A = [a,b[\cup]b,c]$ and $a < b$ and $b < c$ holds $\overline{A} = [a,c]$.

(61) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = \{a\}$ holds $\overline{A} = \{a\}$.

(62) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $A = B$ holds A is open iff B is open.

(63) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]a,+\infty[$ holds A is open.

(64) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty,a[$ holds A is open.

(65) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A =]-\infty,a]$ holds A is closed.

(66) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A = [a,+\infty[$ holds A is closed.

(67) For every real number a holds $[a,+\infty[= \{a\} \cup]a,+\infty[$.

(68) For every real number a holds $]-\infty,a] = \{a\} \cup]-\infty,a[$.

(69) For every real number a holds $]a,+\infty[\subseteq [a,+\infty[$.

(70) For every real number a holds $]-\infty,a[\subseteq]-\infty,a]$.

Let a be a real number. One can check the following observations:

* $]a,+\infty[$ is non empty,
* $]-\infty,a]$ is non empty,
* $]-\infty,a[$ is non empty, and
* $[a,+\infty[$ is non empty.

Next we state a number of propositions:

(71) For every real number a holds $]a,+\infty[\neq \mathbb{R}$.

(72) For every real number a holds $[a,+\infty[\neq \mathbb{R}$.

(73) For every real number a holds $]-\infty,a] \neq \mathbb{R}$.

(74) For every real number a holds $]-\infty,a[\neq\mathbb{R}$.

(75) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A=]a,+\infty[$ holds $\overline{A}=[a,+\infty[$.

(76) For every real number a holds $\overline{]a,+\infty[}=[a,+\infty[$.

(77) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A=]-\infty,a[$ holds $\overline{A}=]-\infty,a]$.

(78) For every real number a holds $\overline{]-\infty,a[}=]-\infty,a]$.

(79) For all subsets A, B of $\mathbb{R}^{\mathbf{1}}$ and for every real number b such that $A=]-\infty,b[$ and $B=]b,+\infty[$ holds A and B are separated.

(80) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a<b$ and $A=[a,b[\cup]b,+\infty[$ holds $\overline{A}=[a,+\infty[$.

(81) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a<b$ and $A=]a,b[\cup]b,+\infty[$ holds $\overline{A}=[a,+\infty[$.

(82) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b, c such that $a<b$ and $b<c$ and $A=]a,b[_{\mathbb{Q}}\cup]b,c[\cup]c,+\infty[$ holds $\overline{A}=[a,+\infty[$.

(83) For every subset A of $\mathbb{R}^{\mathbf{1}}$ holds $A^{\rm c}=\mathbb{R}\setminus A$.

(84) For all real numbers a, b such that $a<b$ holds $]a,b[_{\mathbb{IQ}}$ misses $]a,b[_{\mathbb{Q}}$.

(85) For all real numbers a, b such that $a<b$ holds $\mathbb{R}\setminus]a,b[_{\mathbb{Q}}=]-\infty,a]\cup]a,b[_{\mathbb{IQ}}\cup[b,+\infty[$.

(86) For all real numbers a, b, c such that $a\leq b$ and $b<c$ holds $a\notin]b,c[\cup]c,+\infty[$.

(87) For all real numbers a, b such that $a<b$ holds $b\notin]a,b[\cup]b,+\infty[$.

(88) For all real numbers a, b such that $a<b$ holds $[a,+\infty[\setminus(]a,b[\cup]b,+\infty[)=\{a\}\cup\{b\}$.

(89) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A=]2,3[_{\mathbb{Q}}\cup]3,4[\cup]4,+\infty[$ holds $A^{\rm c}=]-\infty,2]\cup]2,3[_{\mathbb{IQ}}\cup\{3\}\cup\{4\}$.

(90) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A=\{a\}$ holds $A^{\rm c}=]-\infty,a[\cup]a,+\infty[$.

(91) For all real numbers a, b such that $a<b$ holds $]a,+\infty[\cap]-\infty,b]=]a,b]$.

(92) $(]-\infty,1[\cup]1,+\infty[)\cap(]-\infty,2]\cup]2,3[_{\mathbb{IQ}}\cup\{3\}\cup\{4\})=]-\infty,1[\cup]1,2]\cup]2,3[_{\mathbb{IQ}}\cup\{3\}\cup\{4\}$.

(93) For all real numbers a, b such that $a\leq b$ holds $]-\infty,b[\setminus\{a\}=]-\infty,a[\cup]a,b[$.

(94) For all real numbers a, b such that $a\leq b$ holds $]a,+\infty[\setminus\{b\}=]a,b[\cup]b,+\infty[$.

(95) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a\leq b$ and $A=\{a\}\cup[b,+\infty[$ holds $A^{\rm c}=]-\infty,a[\cup]a,b[$.

(96) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a<b$ and $A=]-\infty,a[\cup]a,b[$ holds $\overline{A}=]-\infty,b]$.

(97) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a<b$ and $A=]-\infty,a[\cup]a,b]$ holds $\overline{A}=]-\infty,b]$.

(98) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A=]-\infty,a]$ holds $A^{\rm c}=]a,+\infty[$.

(99) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every real number a such that $A=[a,+\infty[$ holds $A^{\rm c}=]-\infty,a[$.

(100) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b, c such that $a < b$ and $b < c$ and $A =]-\infty, a[\cup]a, b] \cup]b, c[_{\mathbb{IQ}} \cup \{c\}$ holds $\overline{A} =]-\infty, c]$.

(101) Let A be a subset of $\mathbb{R}^{\mathbf{1}}$ and a, b, c, d be real numbers. If $a < b$ and $b < c$ and $A =]-\infty, a[\cup]a, b] \cup]b, c[_{\mathbb{IQ}} \cup \{c\} \cup \{d\}$, then $\overline{A} =]-\infty, c] \cup \{d\}$.

(102) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \leq b$ and $A =]-\infty, a] \cup \{b\}$ holds $A^{\rm c} =]a, b[\cup]b, +\infty[$.

(103) For all real numbers a, b holds $[a, +\infty[\cup \{b\} \neq \mathbb{R}$.

(104) For all real numbers a, b holds $]-\infty, a] \cup \{b\} \neq \mathbb{R}$.

(105) For every topological structure T_1 and for all subsets A, B of T_1 such that $A \neq B$ holds $A^{\rm c} \neq B^{\rm c}$.

(106) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $\mathbb{R} = A^{\rm c}$ holds $A = \emptyset$.

5. Subcontinua of a Real Line

Let us note that $\mathbb{I}$ is arcwise connected.

One can prove the following propositions:

(107) Let X be a compact subset of $\mathbb{R}^{\mathbf{1}}$ and X' be a subset of $\mathbb{R}$. If $X' = X$, then X' is upper bounded and lower bounded.

(108) Let X be a compact subset of $\mathbb{R}^{\mathbf{1}}$, X' be a subset of $\mathbb{R}$, and x be a real number. If $x \in X'$ and $X' = X$, then $\inf X' \leq x$ and $x \leq \sup X'$.

(109) Let C be a non empty compact connected subset of $\mathbb{R}^{\mathbf{1}}$ and C' be a subset of $\mathbb{R}$. If $C = C'$ and $[\inf C', \sup C'] \subseteq C'$, then $[\inf C', \sup C'] = C'$.

(110) Let A be a connected subset of $\mathbb{R}^{\mathbf{1}}$ and a, b, c be real numbers. If $a \leq b$ and $b \leq c$ and $a \in A$ and $c \in A$, then $b \in A$.

(111) For every connected subset A of $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \in A$ and $b \in A$ holds $[a, b] \subseteq A$.

(112) Every non empty compact connected subset of $\mathbb{R}^{\mathbf{1}}$ is a non empty closed-interval subset of $\mathbb{R}$.

(113) For every non empty compact connected subset A of $\mathbb{R}^{\mathbf{1}}$ there exist real numbers a, b such that $a \leq b$ and $A = [a, b]$.

6. Sets with Proper Subsets Only

Let T_1 be a topological structure and let F be a family of subsets of T_1. We say that F has proper subsets if and only if:

(Def. 6) The carrier of $T_1 \notin F$.

The following proposition is true

(114) Let T_1 be a topological structure and F, G be families of subsets of T_1 such that F has proper subsets and $G \subseteq F$. Then G has proper subsets.

Let T_1 be a non empty topological structure. Note that there exists a family of subsets of T_1 which has proper subsets.

The following proposition is true

(115) Let T_1 be a non empty topological structure and A, B be families of subsets of T_1 with proper subsets. Then $A \cup B$ has proper subsets.

Let T be a topological structure and let F be a family of subsets of T. We say that F is open if and only if:

(Def. 7) For every subset P of T such that $P \in F$ holds P is open.

We say that F is closed if and only if:

(Def. 8) For every subset P of T such that $P \in F$ holds P is closed.

Let T be a topological space. Observe that there exists a family of subsets of T which is open, closed, and non empty.

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Received June 12, 2003

Published January 2, 2004