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A Borsuk Theorem on Homotopy Types

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Summary. We present a Borsuk's theorem published first in [1] (compare also [2, pages 119–120]). It is slightly generalized, the assumption of the metrizability is omitted. We introduce concepts needed for the formulation and the proofs of the theorems on upper semi-continuous decompositions, retracts, strong deformation retract. However, only those facts that are necessary in the proof have been proved.

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The articles [20], [8], [22], [11], [23], [5], [21], [19], [17], [13], [12], [3], [7], [16], [6], [15], [24], [10], [9], [14], [4], and [18] provide the notation and terminology for this paper.

1. Preliminaries

We use the following convention: e, u, X, Y, X_1, X_2, Y_1, Y_2 are sets and A is a subset of X.

We now state a number of propositions:

(2)[1] If $e \in [:X_1, Y_1:]$ and $e \in [:X_2, Y_2:]$, then $e \in [:X_1 \cap X_2, Y_1 \cap Y_2:]$.

(3) $(\mathrm{id}_X)^\circ A = A$.

(4) $(\mathrm{id}_X)^{-1}(A) = A$.

(5) For every function F such that $X \subseteq F^{-1}(X_1)$ holds $F^\circ X \subseteq X_1$.

(6) $(X \longmapsto u)^\circ X_1 \subseteq \{u\}$.

(7) If $[:X_1, X_2:] \subseteq [:Y_1, Y_2:]$ and $[:X_1, X_2:] \neq \emptyset$, then $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$.

(9)[2] If $e \subseteq [:X, Y:]$, then $(^\circ\pi_1(X \times Y))(e) = \pi_1(X \times Y)^\circ e$.

(10) If $e \subseteq [:X, Y:]$, then $(^\circ\pi_2(X \times Y))(e) = \pi_2(X \times Y)^\circ e$.

(12)[3] For every subset X_1 of X and for every subset Y_1 of Y such that $[:X_1, Y_1:] \neq \emptyset$ holds $\pi_1(X \times Y)^\circ [:X_1, Y_1:] = X_1$ and $\pi_2(X \times Y)^\circ [:X_1, Y_1:] = Y_1$.

(13) For every subset X_1 of X and for every subset Y_1 of Y such that $[:X_1, Y_1:] \neq \emptyset$ holds $(^\circ\pi_1(X \times Y))([:X_1, Y_1:]) = X_1$ and $(^\circ\pi_2(X \times Y))([:X_1, Y_1:]) = Y_1$.

[1] The proposition (1) has been removed.
[2] The proposition (8) has been removed.
[3] The proposition (11) has been removed.

(14) Let A be a subset of $[:X, Y:]$ and H be a family of subsets of $[:X, Y:]$. Suppose that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [:X_1, Y_1:]$. Then $[:\bigcup((^\circ\pi_1(X \times Y))^\circ H), \bigcap((^\circ\pi_2(X \times Y))^\circ H):] \subseteq A$.

(15) Let A be a subset of $[:X, Y:]$ and H be a family of subsets of $[:X, Y:]$. Suppose that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [:X_1, Y_1:]$. Then $[:\bigcap((^\circ\pi_1(X \times Y))^\circ H), \bigcup((^\circ\pi_2(X \times Y))^\circ H):] \subseteq A$.

(16) Let X be a set, Y be a non empty set, f be a function from X into Y, and H be a family of subsets of X. Then $\bigcup((^\circ f)^\circ H) = f^\circ \bigcup H$.

In the sequel X, Y, Z are non empty sets.
The following propositions are true:

(17) For every set X and for every family a of subsets of X holds $\bigcup\bigcup a = \bigcup\{\bigcup A; A$ ranges over subsets of X: $A \in a\}$.

(18) Let X be a set and D be a family of subsets of X. Suppose $\bigcup D = X$. Let A be a subset of D and B be a subset of X. If $B = \bigcup A$, then $B^{\rm c} \subseteq \bigcup(A^{\rm c})$.

(19) Let F be a function from X into Y and G be a function from X into Z. Suppose that for all elements x, x' of X such that $F(x) = F(x')$ holds $G(x) = G(x')$. Then there exists a function H from Y into Z such that $H \cdot F = G$.

(20) Let given X, Y, Z, y be an element of Y, F be a function from X into Y, and G be a function from Y into Z. Then $F^{-1}(\{y\}) \subseteq (G \cdot F)^{-1}(\{G(y)\})$.

(21) For every function F from X into Y and for every element x of X and for every element z of Z holds $[:F, {\rm id}_Z:](\langle x, z\rangle) = \langle F(x), z\rangle$.

(23)[4] For every function F from X into Y and for every subset A of X and for every subset B of Z holds $[:F, {\rm id}_Z:]^\circ[:A, B:] = [:F^\circ A, B:]$.

(24) Let F be a function from X into Y, y be an element of Y, and z be an element of Z. Then $[:F, {\rm id}_Z:]^{-1}(\{\langle y, z\rangle\}) = [:F^{-1}(\{y\}), \{z\}:]$.

Let B be a non empty set, let A be a set, and let x be an element of B. Then $A \longmapsto x$ is a function from A into B.

2. Partitions

The following propositions are true:

(25) For every family D of subsets of X and for every subset A of D holds $\bigcup A$ is a subset of X.

(26) For every set X and for every partition D of X and for all subsets A, B of D holds $\bigcup(A \cap B) = \bigcup A \cap \bigcup B$.

(27) For every partition D of X and for every subset A of D and for every subset B of X such that $B = \bigcup A$ holds $B^{\rm c} = \bigcup(A^{\rm c})$.

(28) For every equivalence relation E of X holds ${\rm Classes}\, E$ is non empty.

Let X be a non empty set. One can check that there exists a partition of X which is non empty.

Let us consider X and let D be a non empty partition of X. The projection onto D yielding a function from X into D is defined as follows:

(Def. 1) For every element p of X holds $p \in ({\rm the\ projection\ onto\ } D)(p)$.

We now state several propositions:

[4] The proposition (22) has been removed.

(29) Let D be a non empty partition of X, p be an element of X, and A be an element of D. If $p \in A$, then $A =$ (the projection onto $D)(p)$.

(30) For every non empty partition D of X and for every element p of D holds $p =$ (the projection onto $D)^{-1}(\{p\})$.

(31) For every non empty partition D of X and for every subset A of D holds (the projection onto $D)^{-1}(A) = \bigcup A$.

(32) Let D be a non empty partition of X and W be an element of D. Then there exists an element W' of X such that (the projection onto $D)(W') = W$.

(33) Let D be a non empty partition of X and W be a subset of X. Suppose that for every subset B of X such that $B \in D$ and B meets W holds $B \subseteq W$. Then $W =$ (the projection onto $D)^{-1}(($the projection onto $D)^{\circ}W)$.

3. Topological Preliminaries

The following proposition is true

(35)[5] For every topological structure X and for every subspace Y of X holds the carrier of $Y \subseteq$ the carrier of X.

Let X, Y be non empty topological spaces and let F be a map from X into Y. Let us observe that F is continuous if and only if:

(Def. 2) For every point W of X and for every neighbourhood G of $F(W)$ there exists a neighbourhood H of W such that $F^{\circ}H \subseteq G$.

Let X be a 1-sorted structure, let Y be a non empty 1-sorted structure, and let y be an element of Y. The functor $X \longmapsto y$ yielding a map from X into Y is defined by:

(Def. 3) $X \longmapsto y =$ (the carrier of $X) \longmapsto y$.

In the sequel X, Y are non empty topological spaces.

The following proposition is true

(36) For every point y of Y holds $X \longmapsto y$ is continuous.

Let S, T be non empty topological spaces. One can verify that there exists a map from S into T which is continuous.

Let X, Y, Z be non empty topological spaces, let F be a continuous map from X into Y, and let G be a continuous map from Y into Z. Then $G \cdot F$ is a continuous map from X into Z.

Next we state two propositions:

(37) For every continuous map A from X into Y and for every subset G of Y holds $A^{-1}(\operatorname{Int} G) \subseteq \operatorname{Int}(A^{-1}(G))$.

(38) Let W be a point of Y, A be a continuous map from X into Y, and G be a neighbourhood of W. Then $A^{-1}(G)$ is a neighbourhood of $A^{-1}(\{W\})$.

Let X, Y be non empty topological spaces, let W be a point of Y, let A be a continuous map from X into Y, and let G be a neighbourhood of W. Then $A^{-1}(G)$ is a neighbourhood of $A^{-1}(\{W\})$.

We now state three propositions:

(39) Let X be a non empty topological space, A, B be subsets of X, and U_1 be a neighbourhood of B. If $A \subseteq B$, then U_1 is a neighbourhood of A.

(41)[6] For every non empty topological space X and for every point x of X holds $\{x\}$ is compact.

(42) Let X be a topological structure, Y be a subspace of X, A be a subset of X, and B be a subset of Y. If $A = B$, then A is compact iff B is compact.

[5] The proposition (34) has been removed.
[6] The proposition (40) has been removed.

4. Cartesian Product of Topological Spaces

Let X, Y be topological spaces. The functor $[:X, Y:]$ yielding a strict topological space is defined by the conditions (Def. 5).

(Def. 5)[7](i) The carrier of $[:X, Y:] = [:$the carrier of X, the carrier of $Y:]$, and

(ii) the topology of $[:X, Y:] = \{\bigcup A;A$ ranges over families of subsets of $[:X, Y:]$: $A \subseteq \{[:X_1, Y_1:];X_1$ ranges over subsets of X, Y_1 ranges over subsets of Y: $X_1 \in$ the topology of $X \wedge Y_1 \in$ the topology of $Y\}\}$.

Let X, Y be non empty topological spaces. Note that $[:X, Y:]$ is non empty.

Next we state the proposition

(45)[8] Let X, Y be topological spaces and B be a subset of $[:X, Y:]$. Then B is open if and only if there exists a family A of subsets of $[:X, Y:]$ such that $B = \bigcup A$ and for every e such that $e \in A$ there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [:X_1, Y_1:]$ and X_1 is open and Y_1 is open.

Let X, Y be topological spaces, let A be a subset of X, and let B be a subset of Y. Then $[:A, B:]$ is a subset of $[:X, Y:]$.

Let X, Y be non empty topological spaces, let x be a point of X, and let y be a point of Y. Then $\langle x, y\rangle$ is a point of $[:X, Y:]$.

One can prove the following four propositions:

(46) Let X, Y be topological spaces, V be a subset of X, and W be a subset of Y. If V is open and W is open, then $[:V, W:]$ is open.

(47) For all topological spaces X, Y and for every subset V of X and for every subset W of Y holds $\mathrm{Int}[:V, W:] = [:\mathrm{Int}V, \mathrm{Int}W:]$.

(48) Let x be a point of X, y be a point of Y, V be a neighbourhood of x, and W be a neighbourhood of y. Then $[:V, W:]$ is a neighbourhood of $\langle x, y\rangle$.

(49) Let A be a subset of X, B be a subset of Y, V be a neighbourhood of A, and W be a neighbourhood of B. Then $[:V, W:]$ is a neighbourhood of $[:A, B:]$.

Let X, Y be non empty topological spaces, let x be a point of X, let y be a point of Y, let V be a neighbourhood of x, and let W be a neighbourhood of y. Then $[:V, W:]$ is a neighbourhood of $\langle x, y\rangle$.

One can prove the following proposition

(50) For every point X_3 of $[:X, Y:]$ there exists a point W of X and there exists a point T of Y such that $X_3 = \langle W, T\rangle$.

Let X, Y be non empty topological spaces, let A be a subset of X, let t be a point of Y, let V be a neighbourhood of A, and let W be a neighbourhood of t. Then $[:V, W:]$ is a neighbourhood of $[:A, \{t\}:]$.

Let X, Y be topological spaces and let A be a subset of $[:X, Y:]$. The functor $\mathrm{BaseAppr}(A)$ yields a family of subsets of $[:X, Y:]$ and is defined as follows:

(Def. 6) $\mathrm{BaseAppr}(A) = \{[:X_1, Y_1:];X_1$ ranges over subsets of X, Y_1 ranges over subsets of Y: $[:X_1, Y_1:] \subseteq A \wedge X_1$is open $\wedge\, Y_1$is open$\}$.

One can prove the following propositions:

(51) For all topological spaces X, Y and for every subset A of $[:X, Y:]$ holds $\mathrm{BaseAppr}(A)$ is open.

[7] The definition (Def. 4) has been removed.

[8] The propositions (43) and (44) have been removed.

(52) For all topological spaces X, Y and for all subsets A, B of $[:X,Y:]$ such that $A \subseteq B$ holds $\mathrm{BaseAppr}(A) \subseteq \mathrm{BaseAppr}(B)$.

(53) For all topological spaces X, Y and for every subset A of $[:X,Y:]$ holds $\bigcup \mathrm{BaseAppr}(A) \subseteq A$.

(54) For all topological spaces X, Y and for every subset A of $[:X,Y:]$ such that A is open holds $A = \bigcup \mathrm{BaseAppr}(A)$.

(55) For all topological spaces X, Y and for every subset A of $[:X,Y:]$ holds $\mathrm{Int} A = \bigcup \mathrm{BaseAppr}(A)$.

Let X, Y be non empty topological spaces. The functor $\pi_1(X,Y)$ yielding a function from $2^{\text{the carrier of } [:X,Y:]}$ into $2^{\text{the carrier of } X}$ is defined by:

(Def. 7) $\pi_1(X,Y) = {}^\circ\pi_1((\text{the carrier of } X) \times \text{the carrier of } Y)$.

The functor $\pi_2(X,Y)$ yields a function from $2^{\text{the carrier of } [:X,Y:]}$ into $2^{\text{the carrier of } Y}$ and is defined as follows:

(Def. 8) $\pi_2(X,Y) = {}^\circ\pi_2((\text{the carrier of } X) \times \text{the carrier of } Y)$.

The following four propositions are true:

(56) Let A be a subset of $[:X,Y:]$ and H be a family of subsets of $[:X,Y:]$. Suppose that for every e such that $e \in H$ holds $e \subseteq A$ and there exists a subset X_1 of X and there exists a subset Y_1 of Y such that $e = [:X_1, Y_1:]$. Then $[:\bigcup(\pi_1(X,Y)^\circ H), \bigcap(\pi_2(X,Y)^\circ H):] \subseteq A$.

(57) Let H be a family of subsets of $[:X,Y:]$ and C be a set. Suppose $C \in \pi_1(X,Y)^\circ H$. Then there exists a subset D of $[:X,Y:]$ such that $D \in H$ and $C = \pi_1((\text{the carrier of } X) \times \text{the carrier of } Y)^\circ D$.

(58) Let H be a family of subsets of $[:X,Y:]$ and C be a set. Suppose $C \in \pi_2(X,Y)^\circ H$. Then there exists a subset D of $[:X,Y:]$ such that $D \in H$ and $C = \pi_2((\text{the carrier of } X) \times \text{the carrier of } Y)^\circ D$.

(59) Let D be a subset of $[:X,Y:]$. Suppose D is open. Let X_1 be a subset of X and Y_1 be a subset of Y. Then

(i) if $X_1 = \pi_1((\text{the carrier of } X) \times \text{the carrier of } Y)^\circ D$, then X_1 is open, and

(ii) if $Y_1 = \pi_2((\text{the carrier of } X) \times \text{the carrier of } Y)^\circ D$, then Y_1 is open.

Let X, Y be sets, let f be a function from 2^X into 2^Y, and let R be a family of subsets of X. Then $f^\circ R$ is a family of subsets of Y.

Next we state several propositions:

(60) For every family H of subsets of $[:X,Y:]$ such that H is open holds $\pi_1(X,Y)^\circ H$ is open and $\pi_2(X,Y)^\circ H$ is open.

(61) For every family H of subsets of $[:X,Y:]$ such that $\pi_1(X,Y)^\circ H = \emptyset$ or $\pi_2(X,Y)^\circ H = \emptyset$ holds $H = \emptyset$.

(62) Let H be a family of subsets of $[:X,Y:]$, X_1 be a subset of X, and Y_1 be a subset of Y such that H is a cover of $[:X_1, Y_1:]$. Then

(i) if $Y_1 \neq \emptyset$, then $\pi_1(X,Y)^\circ H$ is a cover of X_1, and

(ii) if $X_1 \neq \emptyset$, then $\pi_2(X,Y)^\circ H$ is a cover of Y_1.

(63) Let X, Y be topological spaces, H be a family of subsets of X, and Y be a subset of X. Suppose H is a cover of Y. Then there exists a family F of subsets of X such that $F \subseteq H$ and F is a cover of Y and for every set C such that $C \in F$ holds C meets Y.

(64) Let F be a family of subsets of X and H be a family of subsets of $[:X, Y:]$. Suppose F is finite and $F \subseteq \pi_1(X,Y)^\circ H$. Then there exists a family G of subsets of $[:X, Y:]$ such that $G \subseteq H$ and G is finite and $F = \pi_1(X,Y)^\circ G$.

(65) For every subset X_1 of X and for every subset Y_1 of Y such that $[:X_1, Y_1:] \neq \emptyset$ holds $\pi_1(X,Y)([:X_1, Y_1:]) = X_1$ and $\pi_2(X,Y)([:X_1, Y_1:]) = Y_1$.

(66) $\pi_1(X,Y)(\emptyset) = \emptyset$ and $\pi_2(X,Y)(\emptyset) = \emptyset$.

(67) Let t be a point of Y and A be a subset of X. Suppose A is compact. Let G be a neighbourhood of $[:A, \{t\}:]$. Then there exists a neighbourhood V of A and there exists a neighbourhood W of t such that $[:V, W:] \subseteq G$.

5. Partitions of Topological Spaces

Let X be a 1-sorted structure. The trivial decomposition of X yields a partition of the carrier of X and is defined by:

(Def. 9) The trivial decomposition of $X = \mathrm{Classes}(\mathrm{id}_{\text{the carrier of } X})$.

Let X be a non empty 1-sorted structure. One can verify that the trivial decomposition of X is non empty.

The following proposition is true

(68) For every subset A of X such that $A \in$ the trivial decomposition of X there exists a point x of X such that $A = \{x\}$.

Let X be a topological space and let D be a partition of the carrier of X. The decomposition space of D yields a strict topological space and is defined by the conditions (Def. 10).

(Def. 10)(i) The carrier of the decomposition space of $D = D$, and

(ii) the topology of the decomposition space of $D = \{A;A$ ranges over subsets of D: $\bigcup A \in$ the topology of $X\}$.

Let X be a non empty topological space and let D be a non empty partition of the carrier of X. Note that the decomposition space of D is non empty.

We now state the proposition

(69) Let D be a non empty partition of the carrier of X and A be a subset of D. Then $\bigcup A \in$ the topology of X if and only if $A \in$ the topology of the decomposition space of D.

Let X be a non empty topological space and let D be a non empty partition of the carrier of X. The projection onto D yielding a continuous map from X into the decomposition space of D is defined as follows:

(Def. 11) The projection onto $D =$ the projection onto D.

One can prove the following propositions:

(70) For every non empty partition D of the carrier of X and for every point W of X holds $W \in$ (the projection onto $D)(W)$.

(71) Let D be a non empty partition of the carrier of X and W be a point of the decomposition space of D. Then there exists a point W' of X such that (the projection onto $D)(W') = W$.

(72) Let D be a non empty partition of the carrier of X. Then rng(the projection onto $D) =$ the carrier of the decomposition space of D.

Let X_4 be a non empty topological space, let X be a non empty subspace of X_4, and let D be a non empty partition of the carrier of X. The trivial extension of D yielding a non empty partition of the carrier of X_4 is defined as follows:

(Def. 12) The trivial extension of $D = D \cup \{\{p\}; p$ ranges over points of X_4: $p \notin$ the carrier of $X\}$.

We now state several propositions:

(73) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, and D be a non empty partition of the carrier of X. Then $D \subseteq$ the trivial extension of D.

(74) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and A be a subset of X_4. Suppose $A \in$ the trivial extension of D. Then $A \in D$ or there exists a point x of X_4 such that $x \notin \Omega_X$ and $A = \{x\}$.

(75) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and x be a point of X_4. If $x \notin$ the carrier of X, then $\{x\} \in$ the trivial extension of D.

(76) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and W be a point of X_4. Suppose $W \in$ the carrier of X. Then (the projection onto the trivial extension of $D)(W) =$ (the projection onto $D)(W)$.

(77) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and W be a point of X_4. Suppose $W \notin$ the carrier of X. Then (the projection onto the trivial extension of $D)(W) = \{W\}$.

(78) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and W, W' be points of X_4. Suppose that

(i) $W \notin$ the carrier of X, and

(ii) (the projection onto the trivial extension of $D)(W) =$ (the projection onto the trivial extension of $D)(W')$.

Then $W = W'$.

(79) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and e be a point of X_4. Suppose (the projection onto the trivial extension of $D)(e) \in$ the carrier of the decomposition space of D. Then $e \in$ the carrier of X.

(80) Let X_4 be a non empty topological space, X be a non empty subspace of X_4, D be a non empty partition of the carrier of X, and given e. Suppose $e \in$ the carrier of X. Then (the projection onto the trivial extension of $D)(e) \in$ the carrier of the decomposition space of D.

6. Upper Semicontinuous Decompositions

Let X be a non empty topological space. A non empty partition of the carrier of X is said to be an upper semi-continuous decomposition of X if it satisfies the condition (Def. 13).

(Def. 13) Let A be a subset of X. Suppose $A \in$ it. Let V be a neighbourhood of A. Then there exists a subset W of X such that W is open and $A \subseteq W$ and $W \subseteq V$ and for every subset B of X such that $B \in$ it and B meets W holds $B \subseteq W$.

We now state two propositions:

(81) Let D be an upper semi-continuous decomposition of X, t be a point of the decomposition space of D, and G be a neighbourhood of (the projection onto $D)^{-1}(\{t\})$. Then (the projection onto $D)^{\circ}G$ is a neighbourhood of t.

(82) The trivial decomposition of X is an upper semi-continuous decomposition of X.

Let X be a topological space and let I_1 be a subspace of X. We say that I_1 is closed if and only if:

(Def. 14) For every subset A of X such that $A =$ the carrier of I_1 holds A is closed.

Let X be a topological space. One can check that there exists a subspace of X which is strict and closed.

Let us consider X. Note that there exists a subspace of X which is strict, closed, and non empty.

Let X_4 be a non empty topological space, let X be a closed non empty subspace of X_4, and let D be an upper semi-continuous decomposition of X. Then the trivial extension of D is an upper semi-continuous decomposition of X_4.

Let X be a non empty topological space and let I_1 be an upper semi-continuous decomposition of X. We say that I_1 is upper semi-continuous decomposition-like if and only if:

(Def. 15) For every subset A of X such that $A \in I_1$ holds A is compact.

Let X be a non empty topological space. Observe that there exists an upper semi-continuous decomposition of X which is upper semi-continuous decomposition-like.

Let X be a non empty topological space. An upper semi-continuous decomposition into compacta of X is an upper semi-continuous decomposition-like upper semi-continuous decomposition of X.

Let X_4 be a non empty topological space, let X be a closed non empty subspace of X_4, and let D be an upper semi-continuous decomposition into compacta of X. Then the trivial extension of D is an upper semi-continuous decomposition into compacta of X_4.

Let X be a non empty topological space, let Y be a closed non empty subspace of X, and let D be an upper semi-continuous decomposition into compacta of Y. Then the decomposition space of D is a strict closed subspace of the decomposition space of the trivial extension of D.

7. Borsuk's Theorems on the Decomposition of Retracts

The topological structure $\mathbb{I}$ is defined by the condition (Def. 16).

(Def. 16) Let P be a subset of (the metric space of real numbers)$_{\rm top}$. If $P = [0,1]$, then $\mathbb{I} =$ (the metric space of real numbers)$_{\rm top} \restriction P$.

One can verify that $\mathbb{I}$ is strict, non empty, and topological space-like.

Next we state the proposition

(83) The carrier of $\mathbb{I} = [0,1]$.

The point $0_{\mathbb{I}}$ of $\mathbb{I}$ is defined by:

(Def. 17) $0_{\mathbb{I}} = 0$.

The point $1_{\mathbb{I}}$ of $\mathbb{I}$ is defined by:

(Def. 18) $1_{\mathbb{I}} = 1$.

Let A be a non empty topological space, let B be a non empty subspace of A, and let F be a map from A into B. We say that F is a retraction if and only if:

(Def. 19) For every point W of A such that $W \in$ the carrier of B holds $F(W) = W$.

We introduce F is a retraction as a synonym of F is a retraction.

Let X be a non empty topological space and let Y be a non empty subspace of X. We say that Y is a retract of X if and only if:

(Def. 20) There exists a continuous map from X into Y which is a retraction.

We say that Y is a strong deformation retract of X if and only if the condition (Def. 21) is satisfied.

(Def. 21) There exists a continuous map H from $[:X, \mathbb{I}:]$ into X such that for every point A of X holds $H(\langle A, 0_{\mathbb{I}}\rangle) = A$ and $H(\langle A, 1_{\mathbb{I}}\rangle) \in$ the carrier of Y and if $A \in$ the carrier of Y, then for every point T of $\mathbb{I}$ holds $H(\langle A, T\rangle) = A$.

The following propositions are true:

(84) Let X_4 be a non empty topological space, X be a closed non empty subspace of X_4, and D be an upper semi-continuous decomposition into compacta of X. Suppose X is a retract of X_4. Then the decomposition space of D is a retract of the decomposition space of the trivial extension of D.

(85) Let X_4 be a non empty topological space, X be a closed non empty subspace of X_4, and D be an upper semi-continuous decomposition into compacta of X. Suppose X is a strong deformation retract of X_4. Then the decomposition space of D is a strong deformation retract of the decomposition space of the trivial extension of D.

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