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Birkhoff Theorem for Many Sorted Algebras

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Summary. In this article Birkhoff Variety Theorem for many sorted algebras is proved. A class of algebras is represented by predicate $\mathcal{P}$. Notation $\mathcal{P}[A]$, where A is an algebra, means that A is in class $\mathcal{P}$. All algebras in our class are many sorted over many sorted signature S. The properties of varieties:

- a class $\mathcal{P}$ of algebras is abstract
- a class $\mathcal{P}$ of algebras is closed under subalgebras
- a class $\mathcal{P}$ of algebras is closed under congruences
- a class $\mathcal{P}$ of algebras is closed under products

are published in this paper as:

- for all non-empty algebras A, B over S such that A and B are_isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$
- for every non-empty algebra A over S and for strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$
- for every non-empty algebra A over S and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then$\mathcal{P}[\prod F]$.

This paper is formalization of parts of [21].

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The articles [16], [5], [20], [19], [14], [22], [3], [23], [4], [1], [17], [10], [18], [2], [8], [15], [13], [11], [12], [9], [6], and [7] provide the notation and terminology for this paper.

Let S be a non empty non void many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, let A be a non-empty algebra over S, and let F be a many sorted function from X into the sorts of A. The functor $F^{\#}$ yielding a many sorted function from $\mathrm{Free}(X)$ into A is defined by:

(Def. 1) $F^{\#}$ is a homomorphism of $\mathrm{Free}(X)$ into A and $F^{\#} \restriction \mathrm{FreeGenerator}(X) = F \circ \mathrm{Reverse}(X)$.

Next we state the proposition

(1) Let S be a non empty non void many sorted signature, A be a non-empty algebra over S, X be a non-empty many sorted set indexed by the carrier of S, and F be a many sorted function from X into the sorts of A. Then $\mathrm{rng}_{\kappa} F(\kappa) \subseteq \mathrm{rng}_{\kappa} F^{\#}(\kappa)$.

In this article we present several logical schemes. The scheme *ExFreeAlg 1* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a strict non-empty algebra A over $\mathcal{A}$ and there exists a many sorted function F from $\mathcal{B}$ into A such that
(i) $\mathcal{P}[A]$,
(ii) F is an epimorphism of $\mathcal{B}$ onto A, and
(iii) for every non-empty algebra B over $\mathcal{A}$ and for every many sorted function G from $\mathcal{B}$ into B such that G is a homomorphism of $\mathcal{B}$ into B and $\mathcal{P}[B]$ there exists a many sorted function H from A into B such that H is a homomorphism of A into B and $H \circ F = G$ and for every many sorted function K from A into B such that $K \circ F = G$ holds $H = K$

provided the following requirements are met:

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *ExFreeAlg 2* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a strict non-empty algebra A over $\mathcal{A}$ and there exists a many sorted function F from $\mathcal{B}$ into the sorts of A such that
(i) $\mathcal{P}[A]$, and
(ii) for every non-empty algebra B over $\mathcal{A}$ and for every many sorted function G from $\mathcal{B}$ into the sorts of B such that $\mathcal{P}[B]$ there exists a many sorted function H from A into B such that H is a homomorphism of A into B and $H \circ F = G$ and for every many sorted function K from A into B such that K is a homomorphism of A into B and $K \circ F = G$ holds $H = K$

provided the following conditions are satisfied:

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *Ex hash* deals with a non empty non void many sorted signature $\mathcal{A}$, non-empty algebras $\mathcal{B}$, $\mathcal{C}$ over $\mathcal{A}$, a many sorted function $\mathcal{D}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, a many sorted function $\mathcal{E}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a many sorted function H from $\mathcal{B}$ into $\mathcal{C}$ such that H is a homomorphism of $\mathcal{B}$ into $\mathcal{C}$ and $\mathcal{E}^\# = H \circ \mathcal{D}^\#$

provided the parameters meet the following conditions:

- $\mathcal{P}[\mathcal{C}]$, and
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function h from $\mathcal{B}$ into C such that h is a homomorphism of $\mathcal{B}$ into C and $G = h \circ \mathcal{D}$.

The scheme *EqTerms* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a many sorted function $\mathcal{C}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, a sort symbol $\mathcal{D}$ of $\mathcal{A}$, elements $\mathcal{E}$, $\mathcal{F}$ of the sorts of $\mathrm{T}_{\mathcal{A}}(\mathbb{N})(\mathcal{D})$, and a unary predicate $\mathcal{P}$, and states that:

For every non-empty algebra B over $\mathcal{A}$ such that $\mathcal{P}[B]$ holds $B \models \langle \mathcal{E}, \mathcal{F} \rangle$

provided the parameters meet the following requirements:

- $\mathcal{C}^\#(\mathcal{D})(\mathcal{E}) = \mathcal{C}^\#(\mathcal{D})(\mathcal{F})$, and
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}$) $\longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many

sorted function h from $\mathcal{B}$ into C such that h is a homomorphism of $\mathcal{B}$ into C and $G = h \circ \mathcal{C}$.

The scheme *FreeIsGen* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by the carrier of $\mathcal{A}$, a strict non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a many sorted function $\mathcal{D}$ from $\mathcal{B}$ into the sorts of $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{D}^{\circ}\,\mathcal{B}$ is a non-empty generator set of $\mathcal{C}$

provided the parameters satisfy the following conditions:

- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from $\mathcal{B}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function H from $\mathcal{C}$ into C such that
  (i) H is a homomorphism of $\mathcal{C}$ into C,
  (ii) $H \circ \mathcal{D} = G$, and
  (iii) for every many sorted function K from $\mathcal{C}$ into C such that K is a homomorphism of $\mathcal{C}$ into C and $K \circ \mathcal{D} = G$ holds $H = K$,
- $\mathcal{P}[\mathcal{C}]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$.

The scheme *Hash is onto* deals with a non empty non void many sorted signature $\mathcal{A}$, a strict non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a many sorted function $\mathcal{C}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C}^{\#}$ is an epimorphism of Free((the carrier of $\mathcal{A}) \longmapsto \mathbb{N}$) onto $\mathcal{B}$

provided the parameters satisfy the following conditions:

- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}) \longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{P}[C]$. Then there exists a many sorted function H from $\mathcal{B}$ into C such that
  (i) H is a homomorphism of $\mathcal{B}$ into C,
  (ii) $H \circ \mathcal{C} = G$, and
  (iii) for every many sorted function K from $\mathcal{B}$ into C such that K is a homomorphism of $\mathcal{B}$ into C and $K \circ \mathcal{C} = G$ holds $H = K$,
- $\mathcal{P}[\mathcal{B}]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$.

The scheme *FinGenAlgInVar* deals with a non empty non void many sorted signature $\mathcal{A}$, a strict finitely-generated non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a non-empty algebra $\mathcal{C}$ over $\mathcal{A}$, a many sorted function $\mathcal{D}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{C}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters meet the following conditions:

- $\mathcal{Q}[\mathcal{B}]$,
- $\mathcal{P}[\mathcal{C}]$,
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}) \longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{Q}[C]$. Then there exists a many sorted function h from $\mathcal{C}$ into C such that h is a homomorphism of $\mathcal{C}$ into C and $G = h \circ \mathcal{D}$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$.

The scheme *QuotEpi* deals with a non empty non void many sorted signature $\mathcal{A}$, non-empty algebras $\mathcal{B}$, $\mathcal{C}$ over $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{C}]$

provided the parameters satisfy the following conditions:

- There exists a many sorted function from $\mathcal{B}$ into $\mathcal{C}$ which is an epimorphism of $\mathcal{B}$ onto $\mathcal{C}$,
- $\mathcal{P}[\mathcal{B}]$,

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$.

The scheme *AllFinGen* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters satisfy the following conditions:

- For every strict non-empty finitely-generated subalgebra B of $\mathcal{B}$ holds $\mathcal{P}[B]$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *FreeInModIsInVar 1* deals with a non empty non void many sorted signature $\mathcal{A}$, a non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{Q}[\mathcal{B}]$

provided the following requirements are met:

- Let A be a non-empty algebra over $\mathcal{A}$. Then $\mathcal{Q}[A]$ if and only if for every sort symbol s of $\mathcal{A}$ and for every element e of (the equations of $\mathcal{A})(s)$ such that for every non-empty algebra B over $\mathcal{A}$ such that $\mathcal{P}[B]$ holds $B \models e$ holds $A \models e$, and
- $\mathcal{P}[\mathcal{B}]$.

The scheme *FreeInModIsInVar* deals with a non empty non void many sorted signature $\mathcal{A}$, a strict non-empty algebra $\mathcal{B}$ over $\mathcal{A}$, a many sorted function $\mathcal{C}$ from the carrier of $\mathcal{A} \longmapsto \mathbb{N}$ into the sorts of $\mathcal{B}$, and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\mathcal{P}[\mathcal{B}]$

provided the parameters meet the following requirements:

- Let A be a non-empty algebra over $\mathcal{A}$. Then $\mathcal{Q}[A]$ if and only if for every sort symbol s of $\mathcal{A}$ and for every element e of (the equations of $\mathcal{A})(s)$ such that for every non-empty algebra B over $\mathcal{A}$ such that $\mathcal{P}[B]$ holds $B \models e$ holds $A \models e$,
- Let C be a non-empty algebra over $\mathcal{A}$ and G be a many sorted function from (the carrier of $\mathcal{A}) \longmapsto \mathbb{N}$ into the sorts of C. Suppose $\mathcal{Q}[C]$. Then there exists a many sorted function H from $\mathcal{B}$ into C such that
  - (i) H is a homomorphism of $\mathcal{B}$ into C,
  - (ii) $H \circ \mathcal{C} = G$, and
  - (iii) for every many sorted function K from $\mathcal{B}$ into C such that K is a homomorphism of $\mathcal{B}$ into C and $K \circ \mathcal{C} = G$ holds $H = K$,
- $\mathcal{Q}[\mathcal{B}]$,
- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

The scheme *Birkhoff* deals with a non empty non void many sorted signature $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a set E of equations of $\mathcal{A}$ such that for every non-empty algebra A over $\mathcal{A}$ holds $\mathcal{P}[A]$ iff $A \models E$

provided the following conditions are satisfied:

- For all non-empty algebras A, B over $\mathcal{A}$ such that A and B are isomorphic and $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,

- For every non-empty algebra A over $\mathcal{A}$ and for every strict non-empty subalgebra B of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[B]$,
- For every non-empty algebra A over $\mathcal{A}$ and for every congruence R of A such that $\mathcal{P}[A]$ holds $\mathcal{P}[A/R]$, and
- Let I be a set and F be an algebra family of I over $\mathcal{A}$. Suppose that for every set i such that $i \in I$ there exists an algebra A over $\mathcal{A}$ such that $A = F(i)$ and $\mathcal{P}[A]$. Then $\mathcal{P}[\prod F]$.

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