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Binary Operations

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Summary. In this paper we define binary and unary operations on domains. We also define the following predicates concerning the operations: ... is commutative, ... is associative, ... is the unity of ..., and ... is distributive wrt .... A number of schemes useful in justifying the existence of the operations are proved.

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The articles [4], [3], [5], [6], [1], and [2] provide the notation and terminology for this paper.

Let f be a function and let a, b be sets. The functor $f(a, b)$ yielding a set is defined by:

(Def. 1) $f(a, b) = f(\langle a, b\rangle)$.

In the sequel A is a set.

Let A, B be non empty sets, let C be a set, let f be a function from $[:A, B:]$ into C, let a be an element of A, and let b be an element of B. Then $f(a, b)$ is an element of C.

The following proposition is true

(2)[1] Let A, B, C be non empty sets and f_1, f_2 be functions from $[:A, B:]$ into C. Suppose that for every element a of A and for every element b of B holds $f_1(a, b) = f_2(a, b)$. Then $f_1 = f_2$.

Let A be a set. A unary operation on A is a function from A into A. A binary operation on A is a function from $[:A, A:]$ into A.

We adopt the following convention: u is a unary operation on A, o, o' are binary operations on A, and a, b, c, e, e_1, e_2 are elements of A.

In this article we present several logical schemes. The scheme *BinOpEx* deals with a non empty set $\mathcal{A}$ and a ternary predicate $\mathcal{P}$, and states that:

> There exists a binary operation o on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $\mathcal{P}[a, b, o(a, b)]$

provided the following condition is satisfied:

- For all elements x, y of $\mathcal{A}$ there exists an element z of $\mathcal{A}$ such that $\mathcal{P}[x, y, z]$.

The scheme *BinOpLambda* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

> There exists a binary operation o on $\mathcal{A}$ such that for all elements a, b of $\mathcal{A}$ holds $o(a, b) = \mathcal{F}(a, b)$

for all values of the parameters.

Let us consider A, o. We say that o is commutative if and only if:

(Def. 2) For all a, b holds $o(a, b) = o(b, a)$.

[1] The proposition (1) has been removed.

We say that o is associative if and only if:

(Def. 3) For all a, b, c holds $o(a, o(b, c)) = o(o(a, b), c)$.

We say that o is idempotent if and only if:

(Def. 4) For every a holds $o(a, a) = a$.

Let us mention that every binary operation on $\emptyset$ is empty, associative, and commutative.

Let us consider A, e, o. We say that e is a left unity w.r.t. o if and only if:

(Def. 5) For every a holds $o(e, a) = a$.

We say that e is a right unity w.r.t. o if and only if:

(Def. 6) For every a holds $o(a, e) = a$.

Let us consider A, e, o. We say that e is a unity w.r.t. o if and only if:

(Def. 7) e is a left unity w.r.t. o and a right unity w.r.t. o.

We now state several propositions:

(11)[2] e is a unity w.r.t. o iff for every a holds $o(e, a) = a$ and $o(a, e) = a$.

(12) If o is commutative, then e is a unity w.r.t. o iff for every a holds $o(e, a) = a$.

(13) If o is commutative, then e is a unity w.r.t. o iff for every a holds $o(a, e) = a$.

(14) If o is commutative, then e is a unity w.r.t. o iff e is a left unity w.r.t. o.

(15) If o is commutative, then e is a unity w.r.t. o iff e is a right unity w.r.t. o.

(16) If o is commutative, then e is a left unity w.r.t. o iff e is a right unity w.r.t. o.

(17) If e_1 is a left unity w.r.t. o and e_2 is a right unity w.r.t. o, then $e_1 = e_2$.

(18) If e_1 is a unity w.r.t. o and e_2 is a unity w.r.t. o, then $e_1 = e_2$.

Let us consider A, o. Let us assume that there exists e which is a unity w.r.t. o. The functor $\mathbf{1}_o$ yields an element of A and is defined by:

(Def. 8) $\mathbf{1}_o$ is a unity w.r.t. o.

Let us consider A, o', o. We say that o' is left distributive w.r.t. o if and only if:

(Def. 9) For all a, b, c holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$.

We say that o' is right distributive w.r.t. o if and only if:

(Def. 10) For all a, b, c holds $o'(o(a, b), c) = o(o'(a, c), o'(b, c))$.

Let us consider A, o', o. We say that o' is distributive w.r.t. o if and only if:

(Def. 11) o' is left distributive w.r.t. o and right distributive w.r.t. o.

The following propositions are true:

(23)[3] o' is distributive w.r.t. o iff for all a, b, c holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$ and $o'(o(a, b), c) = o(o'(a, c), o'(b, c))$.

(24) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if for all elements a, b, c of A holds $o'(a, o(b, c)) = o(o'(a, b), o'(a, c))$.

[2] The propositions (3)–(10) have been removed.
[3] The propositions (19)–(22) have been removed.

(25) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if for all elements a, b, c of A holds $o'(o(a,b), c) = o(o'(a,c), o'(b,c))$.

(26) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if o' is left distributive w.r.t. o.

(27) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is distributive w.r.t. o if and only if o' is right distributive w.r.t. o.

(28) Let A be a non empty set and o, o' be binary operations on A. Suppose o' is commutative. Then o' is right distributive w.r.t. o if and only if o' is left distributive w.r.t. o.

Let us consider A, u, o. We say that u is distributive w.r.t. o if and only if:

(Def. 12) For all a, b holds $u(o(a,b)) = o(u(a), u(b))$.

Let A be a non empty set and let o be a binary operation on A. Let us observe that o is commutative if and only if:

(Def. 13) For all elements a, b of A holds $o(a,b) = o(b,a)$.

Let us observe that o is associative if and only if:

(Def. 14) For all elements a, b, c of A holds $o(a, o(b,c)) = o(o(a,b), c)$.

Let us observe that o is idempotent if and only if:

(Def. 15) For every element a of A holds $o(a,a) = a$.

Let A be a non empty set, let e be an element of A, and let o be a binary operation on A. Let us observe that e is a left unity w.r.t. o if and only if:

(Def. 16) For every element a of A holds $o(e,a) = a$.

Let us observe that e is a right unity w.r.t. o if and only if:

(Def. 17) For every element a of A holds $o(a,e) = a$.

Let A be a non empty set and let o', o be binary operations on A. Let us observe that o' is left distributive w.r.t. o if and only if:

(Def. 18) For all elements a, b, c of A holds $o'(a, o(b,c)) = o(o'(a,b), o'(a,c))$.

Let us observe that o' is right distributive w.r.t. o if and only if:

(Def. 19) For all elements a, b, c of A holds $o'(o(a,b), c) = o(o'(a,c), o'(b,c))$.

Let A be a non empty set, let u be a unary operation on A, and let o be a binary operation on A. Let us observe that u is distributive w.r.t. o if and only if:

(Def. 20) For all elements a, b of A holds $u(o(a,b)) = o(u(a), u(b))$.

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