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Binary Arithmetics, Addition and Subtraction of Integers

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Summary. This article is a continuation of [6] and presents the concepts of binary arithmetic operations for integers. There is introduced 2's complement representation of integers and natural numbers to integers are expanded. The binary addition and subtraction for integers are defined and theorems on the relationship between binary and numerical operations presented.

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The articles [8], [11], [1], [3], [9], [12], [7], [4], [2], [5], [10], and [6] provide the notation and terminology for this paper.

Let X be a non empty set, let D be a non empty subset of X, let x, y be sets, and let a, b be elements of D. Then $(x = y \rightarrow a, b)$ is an element of D.

In the sequel i, n are natural numbers and m is a non empty natural number.

Let n be a natural number. The functor $\mathrm{Bin1}(n)$ yields a n-tuple of *Boolean* and is defined as follows:

(Def. 1) For every i such that $i \in \mathrm{Seg}\, n$ holds $(\mathrm{Bin1}(n))_i = (i = 1 \rightarrow \mathit{true}, \mathit{false})$.

Let n be a non empty natural number and let x be a n-tuple of *Boolean*. The functor $\mathrm{Neg2}(x)$ yields a n-tuple of *Boolean* and is defined by:

(Def. 2) $\mathrm{Neg2}(x) = \neg x + \mathrm{Bin1}(n)$.

Let n be a natural number and let x be a n-tuple of *Boolean*. The functor $\mathrm{Intval}(x)$ yields an integer and is defined by:

(Def. 3) $\mathrm{Intval}(x) = \begin{cases} \mathrm{Absval}(x), \text{ if } x_n = \mathit{false}, \\ \mathrm{Absval}(x) - 2^n, \text{ otherwise.} \end{cases}$

Let n be a non empty natural number and let z_1, z_2 be n-tuples of *Boolean*. The functor $\mathrm{Int\_add\_ovfl}(z_1, z_2)$ yielding an element of *Boolean* is defined by:

(Def. 4) $\mathrm{Int\_add\_ovfl}(z_1, z_2) = \neg((z_1)_n) \wedge \neg((z_2)_n) \wedge (\mathrm{carry}(z_1, z_2))_n$.

Let n be a non empty natural number and let z_1, z_2 be n-tuples of *Boolean*. The functor $\mathrm{Int\_add\_udfl}(z_1, z_2)$ yields an element of *Boolean* and is defined by:

(Def. 5) $\mathrm{Int\_add\_udfl}(z_1, z_2) = (z_1)_n \wedge (z_2)_n \wedge \neg((\mathrm{carry}(z_1, z_2))_n)$.

Next we state a number of propositions:

(3)[1] For every 2-tuple z_1 of *Boolean* such that $z_1 = \langle false \rangle \frown \langle false \rangle$ holds $\mathrm{Intval}(z_1) = 0$.

(4) For every 2-tuple z_1 of *Boolean* such that $z_1 = \langle true \rangle \frown \langle false \rangle$ holds $\mathrm{Intval}(z_1) = 1$.

(5) For every 2-tuple z_1 of *Boolean* such that $z_1 = \langle false \rangle \frown \langle true \rangle$ holds $\mathrm{Intval}(z_1) = -2$.

(6) For every 2-tuple z_1 of *Boolean* such that $z_1 = \langle true \rangle \frown \langle true \rangle$ holds $\mathrm{Intval}(z_1) = -1$.

(7) For every i such that $i \in \mathrm{Seg}\, n$ and $i = 1$ holds $(\mathrm{Bin1}(n))_i = true$.

(8) For every i such that $i \in \mathrm{Seg}\, n$ and $i \neq 1$ holds $(\mathrm{Bin1}(n))_i = false$.

(9) $\mathrm{Bin1}(m+1) = (\mathrm{Bin1}(m)) \frown \langle false \rangle$.

(10) For every m holds $\mathrm{Intval}((\mathrm{Bin1}(m)) \frown \langle false \rangle) = 1$.

(11) For every m-tuple z of *Boolean* and for every element d of *Boolean* holds $\neg(z \frown \langle d \rangle) = (\neg z) \frown \langle \neg d \rangle$.

(12) For every m-tuple z of *Boolean* and for every element d of *Boolean* holds $\mathrm{Intval}(z \frown \langle d \rangle) = \mathrm{Absval}(z) - ((d = false \rightarrow 0, 2^m)$ **qua** natural number).

(13) Let z_1, z_2 be m-tuples of *Boolean* and d_1, d_2 be elements of *Boolean*. Then $(\mathrm{Intval}(z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle) + (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = false \rightarrow 0, 2^{m+1})) - (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = false \rightarrow 0, 2^{m+1}) = \mathrm{Intval}(z_1 \frown \langle d_1 \rangle) + \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)$.

(14) Let z_1, z_2 be m-tuples of *Boolean* and d_1, d_2 be elements of *Boolean*. Then $\mathrm{Intval}(z_1 \frown \langle d_1 \rangle + z_2 \frown \langle d_2 \rangle) = ((\mathrm{Intval}(z_1 \frown \langle d_1 \rangle) + \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)) - (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = false \rightarrow 0, 2^{m+1})) + (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, z_2 \frown \langle d_2 \rangle) = false \rightarrow 0, 2^{m+1})$.

(15) For every m and for every m-tuple x of *Boolean* holds $\mathrm{Absval}(\neg x) = (-\mathrm{Absval}(x) + 2^m) - 1$.

(16) For every m-tuple z of *Boolean* and for every element d of *Boolean* holds $\mathrm{Neg2}(z \frown \langle d \rangle) = (\mathrm{Neg2}(z)) \frown \langle \neg d \oplus \mathrm{add\_ovfl}(\neg z, \mathrm{Bin1}(m)) \rangle$.

(17) For every m-tuple z of *Boolean* and for every element d of *Boolean* holds $\mathrm{Intval}(\mathrm{Neg2}(z \frown \langle d \rangle)) + (\mathrm{Int\_add\_ovfl}(\neg(z \frown \langle d \rangle), \mathrm{Bin1}(m+1)) = false \rightarrow 0, 2^{m+1}) = -\mathrm{Intval}(z \frown \langle d \rangle)$.

(18) For every m and for every m-tuple z of *Boolean* and for every element d of *Boolean* holds $\mathrm{Neg2}(\mathrm{Neg2}(z \frown \langle d \rangle)) = z \frown \langle d \rangle$.

Let n be a non empty natural number and let x, y be n-tuples of *Boolean*. The functor $x - y$ yielding a n-tuple of *Boolean* is defined as follows:

(Def. 6) For every i such that $i \in \mathrm{Seg}\, n$ holds $(x - y)_i = x_i \oplus (\mathrm{Neg2}(y))_i \oplus (\mathrm{carry}(x, \mathrm{Neg2}(y)))_i$.

One can prove the following propositions:

(19) For all m-tuples x, y of *Boolean* holds $x - y = x + \mathrm{Neg2}(y)$.

(20) For all m-tuples z_1, z_2 of *Boolean* and for all elements d_1, d_2 of *Boolean* holds $z_1 \frown \langle d_1 \rangle - z_2 \frown \langle d_2 \rangle = (z_1 + \mathrm{Neg2}(z_2)) \frown \langle d_1 \oplus \neg d_2 \oplus \mathrm{add\_ovfl}(\neg z_2, \mathrm{Bin1}(m)) \oplus \mathrm{add\_ovfl}(z_1, \mathrm{Neg2}(z_2)) \rangle$.

(21) Let z_1, z_2 be m-tuples of *Boolean* and d_1, d_2 be elements of *Boolean*. Then $((\mathrm{Intval}(z_1 \frown \langle d_1 \rangle - z_2 \frown \langle d_2 \rangle) + (\mathrm{Int\_add\_ovfl}(z_1 \frown \langle d_1 \rangle, \mathrm{Neg2}(z_2 \frown \langle d_2 \rangle)) = false \rightarrow 0, 2^{m+1})) - (\mathrm{Int\_add\_udfl}(z_1 \frown \langle d_1 \rangle, \mathrm{Neg2}(z_2 \frown \langle d_2 \rangle)) = false \rightarrow 0, 2^{m+1})) + (\mathrm{Int\_add\_ovfl}(\neg(z_2 \frown \langle d_2 \rangle), \mathrm{Bin1}(m+1)) = false \rightarrow 0, 2^{m+1}) = \mathrm{Intval}(z_1 \frown \langle d_1 \rangle) - \mathrm{Intval}(z_2 \frown \langle d_2 \rangle)$.

[1] The propositions (1) and (2) have been removed.

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