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On Some Properties of Real Hilbert Space. Part II

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Summary. This paper is a continuation of our paper [22]. We give an analogue of the necessary and sufficient condition for summable set (i.e. the main theorem of [22]) with respect to summable set by a functional L in real Hilbert space. After presenting certain useful lemmas, we prove our main theorem that the summability for an orthonormal infinite set in real Hilbert space is equivalent to its summability with respect to the square of norm, say $H(x) = (x,x)$. Then we show that the square of norm H commutes with infinite sum operation if the orthonormal set under our consideration is summable. Our main theorem is due to [8].

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The articles [16], [19], [6], [1], [17], [9], [4], [5], [20], [18], [12], [13], [14], [3], [7], [10], [15], [11], [2], [21], and [22] provide the notation and terminology for this paper.

1. Necessary and Sufficient Condition for Summable Set

In this paper X is a real unitary space and x, y are points of X.

The following propositions are true:

(1) Let Y be a subset of X and L be a functional in X. Then Y is summable set by L if and only if for every real number e such that $0 < e$ there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq Y$ and for every finite subset Y_1 of X such that Y_1 is non empty and $Y_1 \subseteq Y$ and Y_0 misses Y_1 holds $|\mathrm{setopfunc}(Y_1, \text{the carrier of } X, \mathbb{R}, L, +_{\mathbb{R}})| < e$.

(2) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite orthogonal family of X. Suppose S is non empty. Let I be a function from the carrier of X into the carrier of X. Suppose $S \subseteq \mathrm{dom} I$ and for every y such that $y \in S$ holds $I(y) = y$. Let H be a function from the carrier of X into $\mathbb{R}$. Suppose $S \subseteq \mathrm{dom} H$ and for every y such that $y \in S$ holds $H(y) = (y|y)$. Then $(\mathrm{setopfunc}(S, \text{the carrier of } X, \text{the carrier of } X, I, \text{the addition of } X) | \mathrm{setopfunc}(S, \text{the carrier of } X, \text{the carrier of } X, I, \text{the addition of } X)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, H, +_{\mathbb{R}})$.

(3) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity. Let S be a finite orthogonal family of X. Suppose S is non empty. Let H be a functional in X. Suppose $S \subseteq \mathrm{dom} H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. Then $(\mathrm{Setsum}(S) | \mathrm{Setsum}(S)) = \mathrm{setopfunc}(S, \text{the carrier of } X, \mathbb{R}, H, +_{\mathbb{R}})$.

(4) Let Y be an orthogonal family of X and Z be a subset of X. If Z is a subset of Y, then Z is an orthogonal family of X.

(5) Let Y be an orthonormal family of X and Z be a subset of X. If Z is a subset of Y, then Z is an orthonormal family of X.

2. Equivalence of Summability

One can prove the following propositions:

(6) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let S be an orthonormal family of X and H be a functional in X. Suppose $S \subseteq \mathrm{dom} H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. Then S is summable_set if and only if S is summable set by H.

(7) Let S be a subset of X. Suppose S is non empty and summable_set. Let e be a real number. Suppose $0 < e$. Then there exists a finite subset Y_0 of X such that Y_0 is non empty and $Y_0 \subseteq S$ and for every finite subset Y_1 of X such that $Y_0 \subseteq Y_1$ and $Y_1 \subseteq S$ holds $|(\mathrm{sum} S | \mathrm{sum} S) - (\mathrm{Setsum}(Y_1) | \mathrm{Setsum}(Y_1))| < e$.

(8) Let given X. Suppose the addition of X is commutative and associative and the addition of X has a unity and X is a Hilbert space. Let S be an orthonormal family of X. Suppose S is non empty. Let H be a functional in X. Suppose $S \subseteq \mathrm{dom} H$ and for every x such that $x \in S$ holds $H(x) = (x|x)$. If S is summable_set, then $(\mathrm{sum} S | \mathrm{sum} S) = \mathrm{SumByfunc}(S, H)$.

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