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Asymptotic Notation. Part I: Theory[1]

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Summary. The widely used textbook by Brassard and Bratley [3] includes a chapter devoted to asymptotic notation (Chapter 3, pp. 79–97). We have attempted to test how suitable the current version of Mizar is for recording this type of material in its entirety. A more detailed report on this experiment will be available separately. This article presents the development of notions and a follow-up article [11] includes examples and solutions to problems. The preliminaries introduce a number of properties of real sequences, some operations on real sequences, and a characterization of convergence. The remaining sections in this article correspond to sections of Chapter 3 of [3]. Section 2 defines the O notation and proves the threshold, maximum, and limit rules. Section 3 introduces the Ω and Θ notations and their analogous rules. Conditional asymptotic notation is defined in Section 4 where smooth functions are also discussed. Section 5 defines some operations on asymptotic notation (we have decided not to introduce the asymptotic notation for functions of several variables as it is a straightforward generalization of notions for unary functions).

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The articles [14], [18], [2], [16], [7], [4], [5], [15], [1], [9], [8], [12], [13], [6], [17], and [10] provide the notation and terminology for this paper.

1. Preliminaries

In this paper c, d denote real numbers and n, N denote natural numbers.

In this article we present several logical schemes. The scheme *FinSegRng1* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a non empty set $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and states that:

$\{\mathcal{F}(i); i$ ranges over natural numbers: $\mathcal{A} \leq i \wedge i \leq \mathcal{B}\}$ is a finite non empty subset of $\mathcal{C}$

provided the following requirement is met:

- $\mathcal{A} \leq \mathcal{B}$.

The scheme *FinImInit1* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

$\{\mathcal{F}(n); n$ ranges over natural numbers: $n \leq \mathcal{A}\}$ is a finite non empty subset of $\mathcal{B}$

for all values of the parameters.

The scheme *FinImInit2* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

$\{\mathcal{F}(n); n$ ranges over natural numbers: $n < \mathcal{A}\}$ is a finite non empty subset of $\mathcal{B}$

provided the parameters have the following property:

- $\mathcal{A} > 0$.

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Let c be a real number. We say that c is logbase if and only if:

(Def. 3)[1] $c > 0$ and $c \neq 1$.

One can check the following observations:

* there exists a real number which is positive,
* there exists a real number which is negative,
* there exists a real number which is logbase,
* there exists a real number which is non negative,
* there exists a real number which is non positive, and
* there exists a real number which is non logbase.

Let f be a sequence of real numbers. We say that f is eventually-nonnegative if and only if:

(Def. 4) There exists N such that for every n such that $n \geq N$ holds $f(n) \geq 0$.

We say that f is positive if and only if:

(Def. 5) For every n holds $f(n) > 0$.

We say that f is eventually-positive if and only if:

(Def. 6) There exists N such that for every n such that $n \geq N$ holds $f(n) > 0$.

We say that f is eventually-nonzero if and only if:

(Def. 7) There exists N such that for every n such that $n \geq N$ holds $f(n) \neq 0$.

We say that f is eventually-nondecreasing if and only if:

(Def. 8) There exists N such that for every n such that $n \geq N$ holds $f(n) \leq f(n+1)$.

Let us observe that there exists a sequence of real numbers which is eventually-nonnegative, eventually-nonzero, positive, eventually-positive, and eventually-nondecreasing.

One can verify the following observations:

* every sequence of real numbers which is positive is also eventually-positive,
* every sequence of real numbers which is eventually-positive is also eventually-nonnegative and eventually-nonzero, and
* every sequence of real numbers which is eventually-nonnegative and eventually-nonzero is also eventually-positive.

Let f, g be eventually-nonnegative sequences of real numbers. Note that $f+g$ is eventually-nonnegative.

Let f be a sequence of real numbers and let c be a real number. The functor $c+f$ yields a sequence of real numbers and is defined by:

(Def. 9) For every n holds $(c+f)(n) = c+f(n)$.

We introduce $f+c$ as a synonym of $c+f$.

Let f be an eventually-nonnegative sequence of real numbers and let c be a positive real number. Observe that $c\,f$ is eventually-nonnegative.

Let f be an eventually-nonnegative sequence of real numbers and let c be a non negative real number. Note that $c+f$ is eventually-nonnegative.

Let f be an eventually-nonnegative sequence of real numbers and let c be a positive real number. One can verify that $c+f$ is eventually-positive.

Let f, g be sequences of real numbers. The functor $\max(f,g)$ yields a sequence of real numbers and is defined as follows:

[1] The definitions (Def. 1) and (Def. 2) have been removed.

(Def. 10) For every n holds $(\max(f,g))(n) = \max(f(n),g(n))$.

Let us notice that the functor $\max(f,g)$ is commutative.

Let f be a sequence of real numbers and let g be an eventually-nonnegative sequence of real numbers. Note that $\max(f,g)$ is eventually-nonnegative.

Let f be a sequence of real numbers and let g be an eventually-positive sequence of real numbers. Note that $\max(f,g)$ is eventually-positive.

Let f, g be sequences of real numbers. We say that g majorizes f if and only if:

(Def. 11) There exists N such that for every n such that $n \geq N$ holds $f(n) \leq g(n)$.

Next we state several propositions:

(1) Let f be a sequence of real numbers and N be a natural number. Suppose that for every n such that $n \geq N$ holds $f(n) \leq f(n+1)$. Let n, m be natural numbers. If $N \leq n$ and $n \leq m$, then $f(n) \leq f(m)$.

(2) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) \neq 0$, then g/f is convergent and $\lim(g/f) = (\lim(f/g))^{-1}$.

(3) For every eventually-nonnegative sequence f of real numbers such that f is convergent holds $0 \leq \lim f$.

(4) Let f, g be sequences of real numbers. If f is convergent and g is convergent and g majorizes f, then $\lim f \leq \lim g$.

(5) Let f be a sequence of real numbers and g be an eventually-nonzero sequence of real numbers. If f/g is divergent to $+\infty$, then g/f is convergent and $\lim(g/f) = 0$.

2. A Notation for "the order of"

Let f be an eventually-nonnegative sequence of real numbers. The functor $O(f)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined as follows:

(Def. 12) $O(f) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,N} (c > 0 \wedge \bigwedge_n (n \geq N \Rightarrow t(n) \leq c \cdot f(n) \wedge t(n) \geq 0))\}$.

We now state a number of propositions:

(6) Let x be a set and f be an eventually-nonnegative sequence of real numbers. Suppose $x \in O(f)$. Then x is an eventually-nonnegative sequence of real numbers.

(7) Let f be a positive sequence of real numbers and t be an eventually-nonnegative sequence of real numbers. Then $t \in O(f)$ if and only if there exists c such that $c > 0$ and for every n holds $t(n) \leq c \cdot f(n)$.

(8) Let f be an eventually-positive sequence of real numbers, t be an eventually-nonnegative sequence of real numbers, and N be a natural number. Suppose $t \in O(f)$ and for every n such that $n \geq N$ holds $f(n) > 0$. Then there exists c such that $c > 0$ and for every n such that $n \geq N$ holds $t(n) \leq c \cdot f(n)$.

(9) For all eventually-nonnegative sequences f, g of real numbers holds $O(f+g) = O(\max(f,g))$.

(10) For every eventually-nonnegative sequence f of real numbers holds $f \in O(f)$.

(11) For all eventually-nonnegative sequences f, g of real numbers such that $f \in O(g)$ holds $O(f) \subseteq O(g)$.

(12) For all eventually-nonnegative sequences f, g, h of real numbers such that $f \in O(g)$ and $g \in O(h)$ holds $f \in O(h)$.

(13) Let f be an eventually-nonnegative sequence of real numbers and c be a positive real number. Then $O(f) = O(c\,f)$.

(14) Let c be a non negative real number and x, f be eventually-nonnegative sequences of real numbers. If $x \in O(f)$, then $x \in O(c+f)$.

(15) For all eventually-positive sequences f, g of real numbers such that f/g is convergent and $\lim(f/g) > 0$ holds $O(f) = O(g)$.

(16) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) = 0$, then $f \in O(g)$ and $g \notin O(f)$.

(17) Let f, g be eventually-positive sequences of real numbers. If f/g is divergent to $+\infty$, then $f \notin O(g)$ and $g \in O(f)$.

3. Other Asymptotic Notation

Let f be an eventually-nonnegative sequence of real numbers. The functor $\Omega(f)$ yields a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ and is defined as follows:

(Def. 13) $\Omega(f) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{d,N}\ (d > 0\ \wedge\ \bigwedge_n\ (n \geq N\ \Rightarrow\ t(n) \geq d \cdot f(n)\ \wedge\ t(n) \geq 0))\}$.

One can prove the following propositions:

(18) Let x be a set and f be an eventually-nonnegative sequence of real numbers. Suppose $x \in \Omega(f)$. Then x is an eventually-nonnegative sequence of real numbers.

(19) For all eventually-nonnegative sequences f, g of real numbers holds $f \in \Omega(g)$ iff $g \in O(f)$.

(20) For every eventually-nonnegative sequence f of real numbers holds $f \in \Omega(f)$.

(21) For all eventually-nonnegative sequences f, g, h of real numbers such that $f \in \Omega(g)$ and $g \in \Omega(h)$ holds $f \in \Omega(h)$.

(22) For all eventually-positive sequences f, g of real numbers such that f/g is convergent and $\lim(f/g) > 0$ holds $\Omega(f) = \Omega(g)$.

(23) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) = 0$, then $g \in \Omega(f)$ and $f \notin \Omega(g)$.

(24) Let f, g be eventually-positive sequences of real numbers. If f/g is divergent to $+\infty$, then $g \notin \Omega(f)$ and $f \in \Omega(g)$.

(25) Let f, t be positive sequences of real numbers. Then $t \in \Omega(f)$ if and only if there exists d such that $d > 0$ and for every n holds $d \cdot f(n) \leq t(n)$.

(26) For all eventually-nonnegative sequences f, g of real numbers holds $\Omega(f+g) = \Omega(\max(f,g))$.

Let f be an eventually-nonnegative sequence of real numbers. The functor $\Theta(f)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined by:

(Def. 14) $\Theta(f) = O(f) \cap \Omega(f)$.

The following propositions are true:

(27) Let f be an eventually-nonnegative sequence of real numbers. Then $\Theta(f) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,d,N}\ (c > 0\ \wedge\ d > 0\ \wedge\ \bigwedge_n\ (n \geq N\ \Rightarrow\ d \cdot f(n) \leq t(n)\ \wedge\ t(n) \leq c \cdot f(n)))\}$.

(28) For every eventually-nonnegative sequence f of real numbers holds $f \in \Theta(f)$.

(29) For all eventually-nonnegative sequences f, g of real numbers such that $f \in \Theta(g)$ holds $g \in \Theta(f)$.

(30) For all eventually-nonnegative sequences f, g, h of real numbers such that $f \in \Theta(g)$ and $g \in \Theta(h)$ holds $f \in \Theta(h)$.

(31) Let f, t be positive sequences of real numbers. Then $t \in \Theta(f)$ if and only if there exist c, d such that $c > 0$ and $d > 0$ and for every n holds $d \cdot f(n) \leq t(n)$ and $t(n) \leq c \cdot f(n)$.

(32) For all eventually-nonnegative sequences f, g of real numbers holds $\Theta(f+g) = \Theta(\max(f,g))$.

(33) For all eventually-positive sequences f, g of real numbers such that f/g is convergent and $\lim(f/g) > 0$ holds $f \in \Theta(g)$.

(34) Let f, g be eventually-positive sequences of real numbers. If f/g is convergent and $\lim(f/g) = 0$, then $f \in O(g)$ and $f \notin \Theta(g)$.

(35) Let f, g be eventually-positive sequences of real numbers. If f/g is divergent to $+\infty$, then $f \in \Omega(g)$ and $f \notin \Theta(g)$.

4. Conditional Asymptotic Notation

Let f be an eventually-nonnegative sequence of real numbers and let X be a set. The functor $O(f|X)$ yields a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ and is defined as follows:

(Def. 15) $O(f|X) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,N} (c > 0 \wedge \bigwedge_n (n \geq N \wedge n \in X \Rightarrow t(n) \leq c \cdot f(n) \wedge t(n) \geq 0))\}$.

Let f be an eventually-nonnegative sequence of real numbers and let X be a set. The functor $\Omega(f|X)$ yielding a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ is defined as follows:

(Def. 16) $\Omega(f|X) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{d,N} (d > 0 \wedge \bigwedge_n (n \geq N \wedge n \in X \Rightarrow t(n) \geq d \cdot f(n) \wedge t(n) \geq 0))\}$.

Let f be an eventually-nonnegative sequence of real numbers and let X be a set. The functor $\Theta(f|X)$ yields a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ and is defined by the condition (Def. 17).

(Def. 17) $\Theta(f|X) = \{t; t$ ranges over elements of $\mathbb{R}^{\mathbb{N}}$: $\bigvee_{c,d,N} (c > 0 \wedge d > 0 \wedge \bigwedge_n (n \geq N \wedge n \in X \Rightarrow d \cdot f(n) \leq t(n) \wedge t(n) \leq c \cdot f(n)))\}$.

The following proposition is true

(36) For every eventually-nonnegative sequence f of real numbers and for every set X holds $\Theta(f|X) = O(f|X) \cap \Omega(f|X)$.

Let f be a sequence of real numbers and let b be a natural number. The functor f_b yielding a sequence of real numbers is defined as follows:

(Def. 18) For every n holds $f_b(n) = f(b \cdot n)$.

Let f be an eventually-nonnegative sequence of real numbers and let b be a natural number. We say that f is smooth w.r.t. b if and only if:

(Def. 19) f is eventually-nondecreasing and $f_b \in O(f)$.

Let f be an eventually-nonnegative sequence of real numbers. We say that f is smooth if and only if:

(Def. 20) For every natural number b such that $b \geq 2$ holds f is smooth w.r.t. b.

One can prove the following propositions:

(37) Let f be an eventually-nonnegative sequence of real numbers. Given a natural number b such that $b \geq 2$ and f is smooth w.r.t. b. Then f is smooth.

(38) Let f be an eventually-nonnegative sequence of real numbers, t be an eventually-nonnegative eventually-nondecreasing sequence of real numbers, and b be a natural number. Suppose f is smooth and $b \geq 2$ and $t \in O(f|\{b^n : n \text{ ranges over natural numbers}\})$. Then $t \in O(f)$.

(39) Let f be an eventually-nonnegative sequence of real numbers, t be an eventually-nonnegative eventually-nondecreasing sequence of real numbers, and b be a natural number. Suppose f is smooth and $b \geq 2$ and $t \in \Omega(f|\{b^n : n \text{ ranges over natural numbers}\})$. Then $t \in \Omega(f)$.

(40) Let f be an eventually-nonnegative sequence of real numbers, t be an eventually-nonnegative eventually-nondecreasing sequence of real numbers, and b be a natural number. Suppose f is smooth and $b \geq 2$ and $t \in \Theta(f|\{b^n : n \text{ ranges over natural numbers}\})$. Then $t \in \Theta(f)$.

5. Operations on Asymptotic Notation

Let X be a non empty set and let F, G be non empty sets of functions from X to $\mathbb{R}$. The functor $F+G$ yields a non empty set of functions from X to $\mathbb{R}$ and is defined by the condition (Def. 21).

(Def. 21) $F+G = \{t;t \text{ ranges over elements of } \mathbb{R}^X\colon \bigvee_{f,g\,:\,\text{element of } \mathbb{R}^X} (f \in F \wedge g \in G \wedge \bigwedge_{n\,:\,\text{element of } X} t(n) = f(n)+g(n))\}$.

Let X be a non empty set and let F, G be non empty sets of functions from X to $\mathbb{R}$. The functor $\max(F,G)$ yielding a non empty set of functions from X to $\mathbb{R}$ is defined by the condition (Def. 22).

(Def. 22) $\max(F,G) = \{t;t \text{ ranges over elements of } \mathbb{R}^X\colon \bigvee_{f,g\,:\,\text{element of } \mathbb{R}^X} (f \in F \wedge g \in G \wedge \bigwedge_{n\,:\,\text{element of } X} t(n) = \max(f(n),g(n)))\}$.

The following propositions are true:

(41) For all eventually-nonnegative sequences f, g of real numbers holds $O(f)+O(g) = O(f+g)$.

(42) For all eventually-nonnegative sequences f, g of real numbers holds $\max(O(f),O(g)) = O(\max(f,g))$.

Let F, G be non empty sets of functions from $\mathbb{N}$ to $\mathbb{R}$. The functor F^G yields a non empty set of functions from $\mathbb{N}$ to $\mathbb{R}$ and is defined by the condition (Def. 23).

(Def. 23) $F^G = \{t;t \text{ ranges over elements of } \mathbb{R}^{\mathbb{N}}\colon \bigvee_{f,g\,:\,\text{element of } \mathbb{R}^{\mathbb{N}}} \bigvee_{N\,:\,\text{element of } \mathbb{N}} (f \in F \wedge g \in G \wedge \bigwedge_{n\,:\,\text{element of } \mathbb{N}} (n \geq N \Rightarrow t(n) = f(n)^{g(n)}))\}$.

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