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Inst. of Computer Science, Univ. of Białystok

Projective Spaces

Wojciech Leończuk
Warsaw University
Białystok

Krzysztof Prażmowski
Warsaw University
Białystok

Summary. In the class of all collinearity structures a subclass of (dimension free) projective spaces, defined by means of a suitable axiom system, is singled out. Whenever a real vector space V is at least 3-dimensional, the structure ProjectiveSpace(V) is a projective space in the above meaning. Some narrower classes of projective spaces are defined: Fano projective spaces, projective planes, and Fano projective planes. For any of the above classes an explicit axiom system is given, as well as an analytical example. There is also a construction a of 3-dimensional and a 4-dimensional real vector space; these are needed to show appropriate examples of projective spaces.

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The articles [8], [12], [10], [2], [3], [1], [7], [11], [9], [5], [6], and [4] provide the notation and terminology for this paper.

For simplicity, we follow the rules: V denotes a real linear space, o, p, q, r, s, u, v, w, y, u_1, v_1, w_1, u_2, v_2, w_2 denote elements of V, a, b, c, d, a_1, b_1, c_1, a_2, c_2 denote real numbers, and z denotes a set.

Next we state several propositions:

(1) Suppose that for all a, b, c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Then

(i) u is a proper vector,

(ii) v is a proper vector,

(iii) w is a proper vector,

(iv) u, v and w are not lineary dependent, and

(v) u and v are not proportional.

(2) Let given u, v, u_1, v_1. Suppose that for all a, b, a_1, b_1 such that $a \cdot u + b \cdot v + a_1 \cdot u_1 + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then u is a proper vector and v is a proper vector and u and v are not proportional and u_1 is a proper vector and v_1 is a proper vector and u_1 and v_1 are not proportional and u, v and u_1 are not lineary dependent and u_1, v_1 and u are not lineary dependent.

(3) Suppose for every w there exist a, b, c such that $w = a \cdot p + b \cdot q + c \cdot r$ and for all a, b, c such that $a \cdot p + b \cdot q + c \cdot r = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$. Let given u, u_1. Then there exists y such that p, q and y are lineary dependent and u, u_1 and y are lineary dependent and y is a proper vector.

(4) Suppose that

(i) for every w there exist a, b, c, d such that $w = a \cdot p + b \cdot q + c \cdot r + d \cdot s$, and

(ii) for all a, b, c, d such that $a \cdot p + b \cdot q + c \cdot r + d \cdot s = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

Let given u, v. Suppose u is a proper vector and v is a proper vector. Then there exist y, w such that

(iii) u, v and w are lineary dependent,

(iv) q, r and y are lineary dependent,

(v) p, w and y are lineary dependent,

(vi) y is a proper vector, and

(vii) w is a proper vector.

(5) Suppose that for all a, b, a_1, b_1 such that $a \cdot u + b \cdot v + a_1 \cdot u_1 + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then there does not exist y such that y is a proper vector and u, v and y are lineary dependent and u_1, v_1 and y are lineary dependent.

Let us consider V, u, v, w. We say that u, v and w are proper vectors if and only if:

(Def. 1) u is a proper vector and v is a proper vector and w is a proper vector.

Let us consider V, u, v, w, u_1, v_1, w_1. We say that u, v, w, u_1, v_1, and w_1 lie on a triangle if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) u, v and w_1 are lineary dependent,

(ii) u, w and v_1 are lineary dependent, and

(iii) v, w and u_1 are lineary dependent.

Let us consider V, o, u, v, w, u_2, v_2, w_2. We say that o, u, v, w, u_2, v_2, and w_2 are perspective if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) o, u and u_2 are lineary dependent,

(ii) o, v and v_2 are lineary dependent, and

(iii) o, w and w_2 are lineary dependent.

Next we state three propositions:

(6) Suppose that

(i) o, u and u_2 are lineary dependent,

(ii) o and u are not proportional,

(iii) o and u_2 are not proportional,

(iv) u and u_2 are not proportional, and

(v) o, u and u_2 are proper vectors.

Then there exist a_1, b_1 such that $b_1 \cdot u_2 = o + a_1 \cdot u$ and $a_1 \neq 0$ and $b_1 \neq 0$ and there exist a_2, c_2 such that $u_2 = c_2 \cdot o + a_2 \cdot u$ and $c_2 \neq 0$ and $a_2 \neq 0$.

(7) Suppose p, q and r are lineary dependent and p and q are not proportional and p, q and r are proper vectors. Then there exist a, b such that $r = a \cdot p + b \cdot q$.

(8) Suppose that o is a proper vector and u, v and w are proper vectors and u_2, v_2 and w_2 are proper vectors and u_1, v_1 and w_1 are proper vectors and o, u, v, w, u_2, v_2, and w_2 are perspective and o and u_2 are not proportional and o and v_2 are not proportional and o and w_2 are not proportional and u and u_2 are not proportional and v and v_2 are not proportional and w and w_2 are not proportional and o, u and v are not lineary dependent and o, u and w are not lineary dependent and o, v and w are not lineary dependent and u, v, w, u_1, v_1, and w_1 lie on a triangle and u_2, v_2, w_2, u_1, v_1, and w_1 lie on a triangle. Then u_1, v_1 and w_1 are lineary dependent.

Let us consider V, o, u, v, w, u_2, v_2, w_2. We say that o, u, v, w, u_2, v_2, and w_2 lie on an angle if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) o, u and u_2 are not lineary dependent,

(ii) o, u and v are lineary dependent,

(iii) o, u and w are lineary dependent,

(iv) o, u_2 and v_2 are lineary dependent, and

(v) o, u_2 and w_2 are lineary dependent.

Let us consider V, o, u, v, w, u_2, v_2, w_2. We say that o, u, v, w, u_2, v_2, w_2 are half-mutually not proportional if and only if the conditions (Def. 5) are satisfied.

(Def. 5) o and v are not proportional and o and w are not proportional and o and v_2 are not proportional and o and w_2 are not proportional and u and v are not proportional and u and w are not proportional and u_2 and v_2 are not proportional and u_2 and w_2 are not proportional and v and w are not proportional and v_2 and w_2 are not proportional.

We now state the proposition

(9) Suppose that o is a proper vector and u, v and w are proper vectors and u_2, v_2 and w_2 are proper vectors and u_1, v_1 and w_1 are proper vectors and o, u, v, w, u_2, v_2, and w_2 lie on an angle and o, u, v, w, u_2, v_2, w_2 are half-mutually not proportional and u, v_2 and w_1 are lineary dependent and u_2, v and w_1 are lineary dependent and u, w_2 and v_1 are lineary dependent and w, u_2 and v_1 are lineary dependent and v, w_2 and u_1 are lineary dependent and w, v_2 and u_1 are lineary dependent. Then u_1, v_1 and w_1 are lineary dependent.

We use the following convention: A denotes a non empty set, f, g, h, f_1 denote elements of $\mathbb{R}^A$, and x_1, x_2, x_3, x_4 denote elements of A.

The following propositions are true:

(10) There exist f, g, h such that

(i) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$,

(ii) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ and if $z \neq x_2$, then $g(z) = 0$, and

(iii) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ and if $z \neq x_3$, then $h(z) = 0$.

(11) Suppose that $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$ and for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ and if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ and if $z \neq x_3$, then $h(z) = 0$. Let given a, b, c. If $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$, then $a = 0$ and $b = 0$ and $c = 0$.

(12) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for all a, b, c if $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$, then $a = 0$ and $b = 0$ and $c = 0$.

(13) Suppose that $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$ and for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ and if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ and if $z \neq x_3$, then $h(z) = 0$. Let h' be an element of $\mathbb{R}^A$. Then there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(14) Suppose $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that for every element h' of $\mathbb{R}^A$ holds there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(15) Suppose $A = \{x_1, x_2, x_3\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_2 \neq x_3$. Then there exist f, g, h such that

(i) for all a, b, c such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$, and

(ii) for every element h' of $\mathbb{R}^A$ there exist a, b, c such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle))$.

(16) There exists a non trivial real linear space V and there exist elements u, v, w of V such that

(i) for all a, b, c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$, and

(ii) for every element y of V there exist a, b, c such that $y = a \cdot u + b \cdot v + c \cdot w$.

(17) There exist f, g, h, f_1 such that

(i) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$,

(ii) for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ and if $z \neq x_2$, then $g(z) = 0$,

(iii) for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ and if $z \neq x_3$, then $h(z) = 0$, and

(iv) for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ and if $z \neq x_4$, then $f_1(z) = 0$.

(18) Suppose that $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_4 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$ and for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ and if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ and if $z \neq x_3$, then $h(z) = 0$ and for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ and if $z \neq x_4$, then $f_1(z) = 0$. Let given a, b, c, d. Suppose $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$. Then $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

(19) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_3 \in A$ and $x_4 \in A$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for all a, b, c, d if $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$, then $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$.

(20) Suppose that $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$ and for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ and if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_2$, then $g(z) = 1$ and if $z \neq x_2$, then $g(z) = 0$ and for every z such that $z \in A$ holds if $z = x_3$, then $h(z) = 1$ and if $z \neq x_3$, then $h(z) = 0$ and for every z such that $z \in A$ holds if $z = x_4$, then $f_1(z) = 1$ and if $z \neq x_4$, then $f_1(z) = 0$. Let h' be an element of $\mathbb{R}^A$. Then there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(21) Suppose $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that for every element h' of $\mathbb{R}^A$ holds there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(22) Suppose $A = \{x_1, x_2, x_3, x_4\}$ and $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_3 \neq x_4$. Then there exist f, g, h, f_1 such that

(i) for all a, b, c, d such that $+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$, and

(ii) for every element h' of $\mathbb{R}^A$ there exist a, b, c, d such that $h' = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f\rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle c, h\rangle)), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle d, f_1\rangle))$.

(23) There exists a non trivial real linear space V and there exist elements u, v, w, u_1 of V such that

(i) for all a, b, c, d such that $a \cdot u + b \cdot v + c \cdot w + d \cdot u_1 = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$, and

(ii) for every element y of V there exist a, b, c, d such that $y = a \cdot u + b \cdot v + c \cdot w + d \cdot u_1$.

Let I_1 be a real linear space. We say that I_1 is up 3-dimensional if and only if:

(Def. 6) There exist elements u, v, w_1 of I_1 such that for all a, b, c such that $a \cdot u + b \cdot v + c \cdot w_1 = 0_{(I_1)}$ holds $a = 0$ and $b = 0$ and $c = 0$.

Let us note that there exists a real linear space which is up 3-dimensional.

One can check that every real linear space which is up 3-dimensional is also non trivial.

Let C_1 be a non empty collinearity structure. Let us observe that C_1 is reflexive if and only if the condition (Def. 7) is satisfied.

(Def. 7) Let p, q, r be elements of C_1. Then p, q and p are collinear and p, p and q are collinear and p, q and q are collinear.

Let us observe that C_1 is transitive if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let p, q, r, r_1, r_2 be elements of C_1. Suppose $p \neq q$ and p, q and r are collinear and p, q and r_1 are collinear and p, q and r_2 are collinear. Then r, r_1 and r_2 are collinear.

Let I_1 be a non empty collinearity structure. We say that I_1 is Vebleian if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let p, p_1, p_2, r, r_1 be elements of I_1. Suppose p, p_1 and r are collinear and p_1, p_2 and r_1 are collinear. Then there exists an element r_2 of I_1 such that p, p_2 and r_2 are collinear and r, r_1 and r_2 are collinear.

We say that I_1 is at least 3 rank if and only if:

(Def. 10) For all elements p, q of I_1 there exists an element r of I_1 such that $p \neq r$ and $q \neq r$ and p, q and r are collinear.

We use the following convention: V denotes a non trivial real linear space, u, v, w, y, u_1, v_1, w_1 denote elements of V, and p, p_1, q, q_1, q_2, q_3, r, r_1, r_2, r_3 denote elements of the projective space over V.

One can prove the following proposition

(24) p, q and r are collinear if and only if there exist u, v, w such that $p =$ the direction of u and $q =$ the direction of v and $r =$ the direction of w and u is a proper vector and v is a proper vector and w is a proper vector and u, v and w are lineary dependent.

Let us consider V. Observe that the projective space over V is reflexive and transitive.

One can prove the following proposition

(25) Suppose p, q and r are collinear. Then

(i) p, r and q are collinear,

(ii) q, p and r are collinear,

(iii) r, q and p are collinear,

(iv) r, p and q are collinear, and

(v) q, r and p are collinear.

Let us consider V. Note that the projective space over V is Vebleian.

Let V be an up 3-dimensional real linear space. Observe that the projective space over V is proper.

Next we state the proposition

(26) Given u, v such that let given a, b. If $a \cdot u + b \cdot v = 0_V$, then $a = 0$ and $b = 0$. Then the projective space over V is at least 3 rank.

Let V be an up 3-dimensional real linear space. Observe that the projective space over V is at least 3 rank.

Let us observe that there exists a non empty collinearity structure which is transitive, reflexive, proper, Vebleian, and at least 3 rank.

A projective space defined in terms of collinearity is a reflexive transitive Vebleian at least 3 rank proper non empty collinearity structure.

Let I_1 be a projective space defined in terms of collinearity. We say that I_1 is Fanoian if and only if the condition (Def. 11) is satisfied.

(Def. 11) Let p_1, r_2, q, r_1, q_1, p, r be elements of I_1. Suppose that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then

(i) p_1, r_2 and q_1 are collinear, or

(ii) p_1, r_2 and r_1 are collinear, or

(iii) p_1, r_1 and q_1 are collinear, or

(iv) r_2, r_1 and q_1 are collinear.

We say that I_1 is Desarguesian if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of I_1. Suppose that $o \neq q_1$ and $p_1 \neq q_1$ and $o \neq q_2$ and $p_2 \neq q_2$ and $o \neq q_3$ and $p_3 \neq q_3$ and o, p_1 and p_2 are not collinear and o, p_1 and p_3 are not collinear and o, p_2 and p_3 are not collinear and p_1, p_2 and r_3 are collinear and q_1, q_2 and r_3 are collinear and p_2, p_3 and r_1 are collinear and q_2, q_3 and r_1 are collinear and p_1, p_3 and r_2 are collinear and q_1, q_3 and r_2 are collinear and o, p_1 and q_1 are collinear and o, p_2 and q_2 are collinear and o, p_3 and q_3 are collinear. Then r_1, r_2 and r_3 are collinear.

We say that I_1 is Pappian if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let o, p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3 be elements of I_1. Suppose that $o \neq p_2$ and $o \neq p_3$ and $p_2 \neq p_3$ and $p_1 \neq p_2$ and $p_1 \neq p_3$ and $o \neq q_2$ and $o \neq q_3$ and $q_2 \neq q_3$ and $q_1 \neq q_2$ and $q_1 \neq q_3$ and o, p_1 and q_1 are not collinear and o, p_1 and p_2 are collinear and o, p_1 and p_3 are collinear and o, q_1 and q_2 are collinear and o, q_1 and q_3 are collinear and p_1, q_2 and r_3 are collinear and q_1, p_2 and r_3 are collinear and p_1, q_3 and r_2 are collinear and p_3, q_1 and r_2 are collinear and p_2, q_3 and r_1 are collinear and p_3, q_2 and r_1 are collinear. Then r_1, r_2 and r_3 are collinear.

Let I_1 be a projective space defined in terms of collinearity. We say that I_1 is 2-dimensional if and only if the condition (Def. 14) is satisfied.

(Def. 14) Let p, p_1, q, q_1 be elements of I_1. Then there exists an element r of I_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

We introduce I_1 is up 3-dimensional as an antonym of I_1 is 2-dimensional.

Let I_1 be a projective space defined in terms of collinearity. We say that I_1 is at most 3 dimensional if and only if the condition (Def. 15) is satisfied.

(Def. 15) Let p, p_1, q, q_1, r_2 be elements of I_1. Then there exist elements r, r_1 of I_1 such that p, q and r are collinear and p_1, q_1 and r_1 are collinear and r_2, r and r_1 are collinear.

Next we state the proposition

(28)[1] Suppose that p_1, r_2 and q are collinear and r_1, q_1 and q are collinear and p_1, r_1 and p are collinear and r_2, q_1 and p are collinear and p_1, q_1 and r are collinear and r_2, r_1 and r are collinear and p, q and r are collinear. Then

(i) p_1, r_2 and q_1 are collinear, or

(ii) p_1, r_2 and r_1 are collinear, or

(iii) p_1, r_1 and q_1 are collinear, or

(iv) r_2, r_1 and q_1 are collinear.

[1] The proposition (27) has been removed.

Let V be an up 3-dimensional real linear space. One can verify that the projective space over V is Fanoian, Desarguesian, and Pappian.

We now state several propositions:

(29) Given u, v, w such that for all a, b, c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = a \cdot u + b \cdot v + c \cdot w$. Then there exist elements x_1, x_2 of the projective space over V such that $x_1 \neq x_2$ and for all r_1, r_2 there exists q such that x_1, x_2 and q are collinear and r_1, r_2 and q are collinear.

(30) Given elements x_1, x_2 of the projective space over V such that $x_1 \neq x_2$ and for all r_1, r_2 there exists q such that x_1, x_2 and q are collinear and r_1, r_2 and q are collinear. Let given p, p_1, q, q_1. Then there exists r such that p, p_1 and r are collinear and q, q_1 and r are collinear.

(31) Given u, v, w such that for all a, b, c such that $a \cdot u + b \cdot v + c \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $c = 0$ and for every y there exist a, b, c such that $y = a \cdot u + b \cdot v + c \cdot w$. Then there exists a projective space C_1 defined in terms of collinearity such that $C_1 =$ the projective space over V and C_1 is 2-dimensional.

(32) Given y, u, v, w such that

(i) for every w_1 there exist a, b, a_1, b_1 such that $w_1 = a \cdot y + b \cdot u + a_1 \cdot v + b_1 \cdot w$, and

(ii) for all a, b, a_1, b_1 such that $a \cdot y + b \cdot u + a_1 \cdot v + b_1 \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then there exist p, q_1, q_2 such that

(iii) p, q_1 and q_2 are not collinear, and

(iv) for all r_1, r_2 there exist q_3, r_3 such that r_1, r_2 and r_3 are collinear and q_1, q_2 and q_3 are collinear and p, r_3 and q_3 are collinear.

(33) Suppose that

(i) the projective space over V is proper and at least 3 rank, and

(ii) there exist p, q_1, q_2 such that p, q_1 and q_2 are not collinear and for all r_1, r_2 there exist q_3, r_3 such that r_1, r_2 and r_3 are collinear and q_1, q_2 and q_3 are collinear and p, r_3 and q_3 are collinear.

Then there exists a projective space C_1 defined in terms of collinearity such that $C_1 =$ the projective space over V and C_1 is at most 3 dimensional.

(34) Given y, u, v, w such that

(i) for every w_1 there exist a, b, c, c_1 such that $w_1 = a \cdot y + b \cdot u + c \cdot v + c_1 \cdot w$, and

(ii) for all a, b, a_1, b_1 such that $a \cdot y + b \cdot u + a_1 \cdot v + b_1 \cdot w = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then there exists a projective space C_1 defined in terms of collinearity such that $C_1 =$ the projective space over V and C_1 is at most 3 dimensional.

(35) Given u, v, u_1, v_1 such that let given a, b, a_1, b_1. If $a \cdot u + b \cdot v + a_1 \cdot u_1 + b_1 \cdot v_1 = 0_V$, then $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$. Then there exists a projective space C_1 defined in terms of collinearity such that $C_1 =$ the projective space over V and C_1 is non 2-dimensional.

(36) Given u, v, u_1, v_1 such that

(i) for every w there exist a, b, a_1, b_1 such that $w = a \cdot u + b \cdot v + a_1 \cdot u_1 + b_1 \cdot v_1$, and

(ii) for all a, b, a_1, b_1 such that $a \cdot u + b \cdot v + a_1 \cdot u_1 + b_1 \cdot v_1 = 0_V$ holds $a = 0$ and $b = 0$ and $a_1 = 0$ and $b_1 = 0$.

Then there exists a projective space C_1 defined in terms of collinearity such that $C_1 =$ the projective space over V and C_1 is up 3-dimensional and at most 3 dimensional.

Let us observe that there exists a projective space defined in terms of collinearity which is strict, Fanoian, Desarguesian, Pappian, and 2-dimensional and there exists a projective space defined in terms of collinearity which is strict, Fanoian, Desarguesian, Pappian, at most 3 dimensional, and up 3-dimensional.

A projective plane defined in terms of collinearity is a 2-dimensional projective space defined in terms of collinearity.

One can prove the following proposition

(37) Let C_1 be a non empty collinearity structure. Then the following statements are equivalent

(i) C_1 is a 2-dimensional projective space defined in terms of collinearity,

(ii) C_1 is an at least 3 rank proper collinearity space and for all elements p, p_1, q, q_1 of C_1 there exists an element r of C_1 such that p, p_1 and r are collinear and q, q_1 and r are collinear.

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