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# Oriented Metric-Affine Plane — Part I

Jarosław Zajkowski
Warsaw University
Białystok

**Summary.** We present (in Euclidean and Minkowskian geometry) definitions and some properties of oriented orthogonality relation. Next we consider consistence Euclidean space and consistence Minkowskian space.

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The articles [6], [1], [2], [8], [7], [4], [3], and [5] provide the notation and terminology for this paper.

Let $V$ be an Abelian non empty loop structure and let $v$, $w$ be elements of $V$. Let us observe that the functor $v+w$ is commutative.

We adopt the following convention: $V$ denotes a real linear space, $u$, $u_1$, $u_2$, $v$, $v_1$, $v_2$, $w$, $w_1$, $x$, $y$ denote vectors of $V$, and $n$ denotes a real number.

Let us consider $V$, $x$, $y$ and let us consider $u$. The functor $\rho^{\mathrm{M}}_{x,y}(u)$ yields a vector of $V$ and is defined by:

(Def. 1) $\rho^{\mathrm{M}}_{x,y}(u) = \pi^1_{x,y}(u) \cdot x + (-\pi^2_{x,y}(u)) \cdot y$.

The following propositions are true:

(1) If $x$, $y$ span the space, then $\rho^{\mathrm{M}}_{x,y}(u+v) = \rho^{\mathrm{M}}_{x,y}(u) + \rho^{\mathrm{M}}_{x,y}(v)$.

(2) If $x$, $y$ span the space, then $\rho^{\mathrm{M}}_{x,y}(n \cdot u) = n \cdot \rho^{\mathrm{M}}_{x,y}(u)$.

(3) If $x$, $y$ span the space, then $\rho^{\mathrm{M}}_{x,y}(0_V) = 0_V$.

(4) If $x$, $y$ span the space, then $\rho^{\mathrm{M}}_{x,y}(-u) = -\rho^{\mathrm{M}}_{x,y}(u)$.

(5) If $x$, $y$ span the space, then $\rho^{\mathrm{M}}_{x,y}(u-v) = \rho^{\mathrm{M}}_{x,y}(u) - \rho^{\mathrm{M}}_{x,y}(v)$.

(6) If $x$, $y$ span the space and $\rho^{\mathrm{M}}_{x,y}(u) = \rho^{\mathrm{M}}_{x,y}(v)$, then $u = v$.

(7) If $x$, $y$ span the space, then $\rho^{\mathrm{M}}_{x,y}(\rho^{\mathrm{M}}_{x,y}(u)) = u$.

(8) If $x$, $y$ span the space, then there exists $v$ such that $u = \rho^{\mathrm{M}}_{x,y}(v)$.

Let us consider $V$, $x$, $y$ and let us consider $u$. The functor $\rho^{\mathrm{E}}_{x,y}(u)$ yields a vector of $V$ and is defined by:

(Def. 2) $\rho^{\mathrm{E}}_{x,y}(u) = \pi^2_{x,y}(u) \cdot x + (-\pi^1_{x,y}(u)) \cdot y$.

Next we state several propositions:

(9) If $x$, $y$ span the space, then $\rho^{\mathrm{E}}_{x,y}(-v) = -\rho^{\mathrm{E}}_{x,y}(v)$.

(10) If $x$, $y$ span the space, then $\rho^{\mathrm{E}}_{x,y}(u+v) = \rho^{\mathrm{E}}_{x,y}(u) + \rho^{\mathrm{E}}_{x,y}(v)$.

(11) If $x$, $y$ span the space, then $\rho^{\mathrm{E}}_{x,y}(u-v) = \rho^{\mathrm{E}}_{x,y}(u) - \rho^{\mathrm{E}}_{x,y}(v)$.

(12) If $x$, $y$ span the space, then $\rho^{\mathrm{E}}_{x,y}(n \cdot u) = n \cdot \rho^{\mathrm{E}}_{x,y}(u)$.

(13) If $x$, $y$ span the space and $\rho^{\mathrm{E}}_{x,y}(u) = \rho^{\mathrm{E}}_{x,y}(v)$, then $u = v$.

(14) If $x$, $y$ span the space, then $\rho^{\mathrm{E}}_{x,y}(\rho^{\mathrm{E}}_{x,y}(u)) = -u$.

(15) If $x$, $y$ span the space, then there exists $v$ such that $\rho^{\mathrm{E}}_{x,y}(v) = u$.

Let us consider $V$ and let us consider $x$, $y$, $u$, $v$, $u_1$, $v_1$. We say that the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$ if and only if:

(Def. 3) $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upuparrows u_1, v_1$.

We say that the segments $u$, $v$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$ if and only if:

(Def. 4) $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upuparrows u_1, v_1$.

We now state a number of propositions:

(16) If $x$, $y$ span the space, then if $u, v \upuparrows u_1, v_1$, then $\rho^{\mathrm{E}}_{x,y}(u), \rho^{\mathrm{E}}_{x,y}(v) \upuparrows \rho^{\mathrm{E}}_{x,y}(u_1), \rho^{\mathrm{E}}_{x,y}(v_1)$.

(17) If $x$, $y$ span the space, then if $u, v \upuparrows u_1, v_1$, then $\rho^{\mathrm{M}}_{x,y}(u), \rho^{\mathrm{M}}_{x,y}(v) \upuparrows \rho^{\mathrm{M}}_{x,y}(u_1), \rho^{\mathrm{M}}_{x,y}(v_1)$.

(18) Suppose $x$, $y$ span the space. Suppose the segments $u$, $u_1$ and $v$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$. Then the segments $v$, $v_1$ and $u_1$, $u$ are E-coherently orthogonal in the basis $x$, $y$.

(19) Suppose $x$, $y$ span the space. Suppose the segments $u$, $u_1$ and $v$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$. Then the segments $v$, $v_1$ and $u$, $u_1$ are M-coherently orthogonal in the basis $x$, $y$.

(20) The segments $u$, $u$ and $v$, $w$ are E-coherently orthogonal in the basis $x$, $y$.

(21) The segments $u$, $u$ and $v$, $w$ are M-coherently orthogonal in the basis $x$, $y$.

(22) The segments $u$, $v$ and $w$, $w$ are E-coherently orthogonal in the basis $x$, $y$.

(23) The segments $u$, $v$ and $w$, $w$ are M-coherently orthogonal in the basis $x$, $y$.

(24) If $x$, $y$ span the space, then $u$, $v$, $\rho^{\mathrm{E}}_{x,y}(u)$ and $\rho^{\mathrm{E}}_{x,y}(v)$ are orthogonal w.r.t. $x$, $y$.

(25) The segments $u$, $v$ and $\rho^{\mathrm{E}}_{x,y}(u)$, $\rho^{\mathrm{E}}_{x,y}(v)$ are E-coherently orthogonal in the basis $x$, $y$.

(26) The segments $u$, $v$ and $\rho^{\mathrm{M}}_{x,y}(u)$, $\rho^{\mathrm{M}}_{x,y}(v)$ are M-coherently orthogonal in the basis $x$, $y$.

(27) Suppose $x$, $y$ span the space. Then $u, v \upuparrows u_1, v_1$ if and only if there exist $u_2$, $v_2$ such that $u_2 \neq v_2$ and the segments $u_2$, $v_2$ and $u$, $v$ are E-coherently orthogonal in the basis $x$, $y$ and the segments $u_2$, $v_2$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$.

(28) Suppose $x$, $y$ span the space. Then $u, v \upuparrows u_1, v_1$ if and only if there exist $u_2$, $v_2$ such that $u_2 \neq v_2$ and the segments $u_2$, $v_2$ and $u$, $v$ are M-coherently orthogonal in the basis $x$, $y$ and the segments $u_2$, $v_2$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$.

(29) Suppose $x$, $y$ span the space. Then $u$, $v$, $u_1$ and $v_1$ are orthogonal w.r.t. $x$, $y$ if and only if one of the following conditions is satisfied:

(i) the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$, or

(ii) the segments $u$, $v$ and $v_1$, $u_1$ are E-coherently orthogonal in the basis $x$, $y$.

(30) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u$, $v$ and $v_1$, $u_1$ are E-coherently orthogonal in the basis $x$, $y$.

Then $u = v$ or $u_1 = v_1$.

(31) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $v$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u$, $v$ and $v_1$, $u_1$ are M-coherently orthogonal in the basis $x$, $y$.

Then $u = v$ or $u_1 = v_1$.

(32) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u$, $v$ and $u_1$, $w$ are E-coherently orthogonal in the basis $x$, $y$.

Then

(iv) the segments $u$, $v$ and $v_1$, $w$ are E-coherently orthogonal in the basis $x$, $y$, or

(v) the segments $u$, $v$ and $w$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$.

(33) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $v$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u$, $v$ and $u_1$, $w$ are M-coherently orthogonal in the basis $x$, $y$.

Then

(iv) the segments $u$, $v$ and $v_1$, $w$ are M-coherently orthogonal in the basis $x$, $y$, or

(v) the segments $u$, $v$ and $w$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$.

(34) Suppose the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$. Then the segments $v$, $u$ and $v_1$, $u_1$ are E-coherently orthogonal in the basis $x$, $y$.

(35) Suppose the segments $u$, $v$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$. Then the segments $v$, $u$ and $v_1$, $u_1$ are M-coherently orthogonal in the basis $x$, $y$.

(36) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u$, $v$ and $v_1$, $w$ are E-coherently orthogonal in the basis $x$, $y$.

Then the segments $u$, $v$ and $u_1$, $w$ are E-coherently orthogonal in the basis $x$, $y$.

(37) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $v$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u$, $v$ and $v_1$, $w$ are M-coherently orthogonal in the basis $x$, $y$.

Then the segments $u$, $v$ and $u_1$, $w$ are M-coherently orthogonal in the basis $x$, $y$.

(38) Suppose $x$, $y$ span the space. Let given $u$, $v$, $w$. Then there exists $u_1$ such that $w \neq u_1$ and the segments $w$, $u_1$ and $u$, $v$ are E-coherently orthogonal in the basis $x$, $y$.

(39) Suppose $x$, $y$ span the space. Let given $u$, $v$, $w$. Then there exists $u_1$ such that $w \neq u_1$ and the segments $w$, $u_1$ and $u$, $v$ are M-coherently orthogonal in the basis $x$, $y$.

(40) Suppose $x$, $y$ span the space. Let given $u$, $v$, $w$. Then there exists $u_1$ such that $w \neq u_1$ and the segments $u$, $v$ and $w$, $u_1$ are E-coherently orthogonal in the basis $x$, $y$.

(41) Suppose $x$, $y$ span the space. Let given $u$, $v$, $w$. Then there exists $u_1$ such that $w \neq u_1$ and the segments $u$, $v$ and $w$, $u_1$ are M-coherently orthogonal in the basis $x$, $y$.

(42) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $u_1$ and $v$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$,

(iii) the segments $w$, $w_1$ and $v$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$, and

(iv) the segments $w$, $w_1$ and $u_2$, $v_2$ are E-coherently orthogonal in the basis $x$, $y$.

Then $w = w_1$ or $v = v_1$ or the segments $u$, $u_1$ and $u_2$, $v_2$ are E-coherently orthogonal in the basis $x$, $y$.

(43) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $u_1$ and $v$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$,

(iii) the segments $w$, $w_1$ and $v$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$, and

(iv) the segments $w$, $w_1$ and $u_2$, $v_2$ are M-coherently orthogonal in the basis $x$, $y$.

Then $w = w_1$ or $v = v_1$ or the segments $u$, $u_1$ and $u_2$, $v_2$ are M-coherently orthogonal in the basis $x$, $y$.

(46)[1] Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $u_1$ and $v$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$,

(iii) the segments $v$, $v_1$ and $w$, $w_1$ are E-coherently orthogonal in the basis $x$, $y$, and

(iv) the segments $u_2$, $v_2$ and $w$, $w_1$ are E-coherently orthogonal in the basis $x$, $y$.

Then the segments $u$, $u_1$ and $u_2$, $v_2$ are E-coherently orthogonal in the basis $x$, $y$ or $v = v_1$ or $w = w_1$.

(47) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $u_1$ and $v$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$,

(iii) the segments $v$, $v_1$ and $w$, $w_1$ are M-coherently orthogonal in the basis $x$, $y$, and

(iv) the segments $u_2$, $v_2$ and $w$, $w_1$ are M-coherently orthogonal in the basis $x$, $y$.

Then the segments $u$, $u_1$ and $u_2$, $v_2$ are M-coherently orthogonal in the basis $x$, $y$ or $v = v_1$ or $w = w_1$.

(48) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $u_1$ and $v$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$,

(iii) the segments $v$, $v_1$ and $w$, $w_1$ are E-coherently orthogonal in the basis $x$, $y$, and

(iv) the segments $u$, $u_1$ and $u_2$, $v_2$ are E-coherently orthogonal in the basis $x$, $y$.

Then the segments $u_2$, $v_2$ and $w$, $w_1$ are E-coherently orthogonal in the basis $x$, $y$ or $v = v_1$ or $u = u_1$.

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[1] The propositions (44) and (45) have been removed.

(49) Suppose that

(i) $x$, $y$ span the space,

(ii) the segments $u$, $u_1$ and $v$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$,

(iii) the segments $v$, $v_1$ and $w$, $w_1$ are M-coherently orthogonal in the basis $x$, $y$, and

(iv) the segments $u$, $u_1$ and $u_2$, $v_2$ are M-coherently orthogonal in the basis $x$, $y$.

Then the segments $u_2$, $v_2$ and $w$, $w_1$ are M-coherently orthogonal in the basis $x$, $y$ or $v = v_1$ or $u = u_1$.

(50) Suppose $x$, $y$ span the space. Let given $v$, $w$, $u_1$, $v_1$, $w_1$. Suppose that

(i) the segments $v$, $v_1$ and $w$, $u_1$ are not E-coherently orthogonal in the basis $x$, $y$,

(ii) the segments $v$, $v_1$ and $u_1$, $w$ are not E-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u_1$, $w_1$ and $u_1$, $w$ are E-coherently orthogonal in the basis $x$, $y$.

Then there exists $u_2$ such that

(iv) the segments $v$, $v_1$ and $v$, $u_2$ are E-coherently orthogonal in the basis $x$, $y$ or the segments $v$, $v_1$ and $u_2$, $v$ are E-coherently orthogonal in the basis $x$, $y$, and

(v) the segments $u_1$, $w_1$ and $u_1$, $u_2$ are E-coherently orthogonal in the basis $x$, $y$ or the segments $u_1$, $w_1$ and $u_2$, $u_1$ are E-coherently orthogonal in the basis $x$, $y$.

(51) Suppose $x$, $y$ span the space. Then there exist $u$, $v$, $w$ such that

(i) the segments $u$, $v$ and $u$, $w$ are E-coherently orthogonal in the basis $x$, $y$, and

(ii) for all $v_1$, $w_1$ such that the segments $v_1$, $w_1$ and $u$, $v$ are E-coherently orthogonal in the basis $x$, $y$ holds the segments $v_1$, $w_1$ and $u$, $w$ are not E-coherently orthogonal in the basis $x$, $y$ and the segments $v_1$, $w_1$ and $w$, $u$ are not E-coherently orthogonal in the basis $x$, $y$ or $v_1 = w_1$.

(52) Suppose $x$, $y$ span the space. Let given $v$, $w$, $u_1$, $v_1$, $w_1$. Suppose that

(i) the segments $v$, $v_1$ and $w$, $u_1$ are not M-coherently orthogonal in the basis $x$, $y$,

(ii) the segments $v$, $v_1$ and $u_1$, $w$ are not M-coherently orthogonal in the basis $x$, $y$, and

(iii) the segments $u_1$, $w_1$ and $u_1$, $w$ are M-coherently orthogonal in the basis $x$, $y$.

Then there exists $u_2$ such that

(iv) the segments $v$, $v_1$ and $v$, $u_2$ are M-coherently orthogonal in the basis $x$, $y$ or the segments $v$, $v_1$ and $u_2$, $v$ are M-coherently orthogonal in the basis $x$, $y$, and

(v) the segments $u_1$, $w_1$ and $u_1$, $u_2$ are M-coherently orthogonal in the basis $x$, $y$ or the segments $u_1$, $w_1$ and $u_2$, $u_1$ are M-coherently orthogonal in the basis $x$, $y$.

(53) Suppose $x$, $y$ span the space. Then there exist $u$, $v$, $w$ such that

(i) the segments $u$, $v$ and $u$, $w$ are M-coherently orthogonal in the basis $x$, $y$, and

(ii) for all $v_1$, $w_1$ such that the segments $v_1$, $w_1$ and $u$, $v$ are M-coherently orthogonal in the basis $x$, $y$ holds the segments $v_1$, $w_1$ and $u$, $w$ are not M-coherently orthogonal in the basis $x$, $y$ and the segments $v_1$, $w_1$ and $w$, $u$ are not M-coherently orthogonal in the basis $x$, $y$ or $v_1 = w_1$.

In the sequel $u_3$, $v_3$ are sets.

Let us consider $V$ and let us consider $x$, $y$. The Euclidean oriented orthogonality defined over $V$, $x$, $y$ yields a binary relation on $[:$ the carrier of $V$, the carrier of $V :]$ and is defined by the condition (Def. 5).

(Def. 5) The following statements are equivalent

(i) $\langle u_3, v_3 \rangle \in$ the Euclidean oriented orthogonality defined over $V$, $x$, $y$,

(ii) there exist $u_1$, $u_2$, $v_1$, $v_2$ such that $u_3 = \langle u_1, u_2 \rangle$ and $v_3 = \langle v_1, v_2 \rangle$ and the segments $u_1$, $u_2$ and $v_1$, $v_2$ are E-coherently orthogonal in the basis $x$, $y$.

Let us consider $V$ and let us consider $x$, $y$. The Minkowskian oriented orthogonality defined over $V$, $x$, $y$ yielding a binary relation on $[:$the carrier of $V$, the carrier of $V:]$ is defined by the condition (Def. 6).

(Def. 6) The following statements are equivalent

(i) $\langle u_3, v_3\rangle \in$ the Minkowskian oriented orthogonality defined over $V$, $x$, $y$,

(ii) there exist $u_1$, $u_2$, $v_1$, $v_2$ such that $u_3 = \langle u_1, u_2\rangle$ and $v_3 = \langle v_1, v_2\rangle$ and the segments $u_1$, $u_2$ and $v_1$, $v_2$ are M-coherently orthogonal in the basis $x$, $y$.

Let us consider $V$ and let us consider $x$, $y$. The functor CESpace$(V,x,y)$ yields a strict affine structure and is defined by:

(Def. 7) CESpace$(V,x,y) = \langle$the carrier of $V$, the Euclidean oriented orthogonality defined over $V$, $x$, $y\rangle$.

Let us consider $V$ and let us consider $x$, $y$. One can check that CESpace$(V,x,y)$ is non empty.

Let us consider $V$ and let us consider $x$, $y$. The functor CMSpace$(V,x,y)$ yields a strict affine structure and is defined by:

(Def. 8) CMSpace$(V,x,y) = \langle$the carrier of $V$, the Minkowskian oriented orthogonality defined over $V$, $x$, $y\rangle$.

Let us consider $V$ and let us consider $x$, $y$. Observe that CMSpace$(V,x,y)$ is non empty.

We now state two propositions:

(54) $u_3$ is an element of CESpace$(V,x,y)$ iff $u_3$ is a vector of $V$.

(55) $u_3$ is an element of CMSpace$(V,x,y)$ iff $u_3$ is a vector of $V$.

In the sequel $p$, $q$, $r$, $s$ are elements of CESpace$(V,x,y)$.

The following proposition is true

(56) Suppose $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$. Then $p,q \rceil\!\!\rceil r,s$ if and only if the segments $u$, $v$ and $u_1$, $v_1$ are E-coherently orthogonal in the basis $x$, $y$.

In the sequel $p$, $q$, $r$, $s$ are elements of CMSpace$(V,x,y)$.

Next we state the proposition

(57) Suppose $u = p$ and $v = q$ and $u_1 = r$ and $v_1 = s$. Then $p,q \rceil\!\!\rceil r,s$ if and only if the segments $u$, $v$ and $u_1$, $v_1$ are M-coherently orthogonal in the basis $x$, $y$.

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