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From Double Loops to Fields[1]

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Summary. This paper contains the second part of the set of articles concerning the theory of algebraic structures, based on [4, pp. 9-12] (pages 4–6 of the English edition).

First the basic structure $\langle F, +, \cdot, 1, 0\rangle$ is defined. Following it the consecutive structures are contemplated in details, including double loop, left quasi-field, right quasi-field, double sided quasi-field, skew field and field. These structures are created by gradually augmenting the basic structure with new axioms of commutativity, associativity, distributivity etc. Each part of the article begins with the set of auxiliary theorems related to the structure under consideration, that are necessary for the existence proof of each defined mode. Next the mode and proof of its existence is included. If the current set of axioms may be replaced with a different and equivalent one, the appropriate proof is performed following the definition of that mode. With the introduction of double loop the "-" function is defined. Some interesting features of this function are also included.

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The articles [5], [1], [2], and [3] provide the notation and terminology for this paper.

In this paper L is a non empty double loop structure.

Let us observe that $\mathbb{R}_{\mathrm{F}}$ is multiplicative loop with zero-like.

Let L be an add-left-cancelable add-right-invertible non empty loop structure and let a be an element of L. The functor $-a$ yielding an element of L is defined by:

(Def. 7)[1] $a + -a = 0_L$.

Let L be an add-left-cancelable add-right-invertible non empty loop structure and let a, b be elements of L. The functor $a - b$ yielding an element of L is defined as follows:

(Def. 8) $a - b = a + -b$.

One can verify that there exists a non empty double loop structure which is strict, Abelian, add-associative, commutative, associative, distributive, non degenerated, left zeroed, right zeroed, loop-like, well unital, and multiplicative loop with zero-like.

A double loop is a left zeroed right zeroed loop-like well unital multiplicative loop with zero-like non empty double loop structure.

A left quasi-field is an Abelian add-associative right distributive non degenerated double loop.

In the sequel a, b, c, x, y denote elements of L.

One can prove the following propositions:

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[1] The definitions (Def. 1)–(Def. 6) have been removed.

(12)[2] L is a left quasi-field if and only if the following conditions are satisfied:

for every a holds $a+0_L=a$ and for every a there exists x such that $a+x=0_L$ and for all a, b, c holds $(a+b)+c=a+(b+c)$ and for all a, b holds $a+b=b+a$ and $0_L \neq \mathbf{1}_L$ and for every a holds $a\cdot\mathbf{1}_L=a$ and for every a holds $\mathbf{1}_L\cdot a=a$ and for all a, b such that $a\neq 0_L$ there exists x such that $a\cdot x=b$ and for all a, b such that $a\neq 0_L$ there exists x such that $x\cdot a=b$ and for all a, x, y such that $a\neq 0_L$ holds if $a\cdot x=a\cdot y$, then $x=y$ and for all a, x, y such that $a\neq 0_L$ holds if $x\cdot a=y\cdot a$, then $x=y$ and for every a holds $a\cdot 0_L=0_L$ and for every a holds $0_L\cdot a=0_L$ and for all a, b, c holds $a\cdot(b+c)=a\cdot b+a\cdot c$.

(14)[3] For every Abelian right distributive double loop G and for all elements a, b of G holds $a\cdot -b=-a\cdot b$.

(15) Let G be an Abelian add-left-cancelable add-right-invertible non empty loop structure and a be an element of G. Then $--a=a$.

(16) For every Abelian right distributive double loop G holds $(-\mathbf{1}_G)\cdot -\mathbf{1}_G=\mathbf{1}_G$.

(17) For every Abelian right distributive double loop G and for all elements a, x, y of G holds $a\cdot(x-y)=a\cdot x-a\cdot y$.

A right quasi-field is an Abelian add-associative left distributive non degenerated double loop.
Next we state the proposition

(19)[4] L is a right quasi-field if and only if the following conditions are satisfied:

for every a holds $a+0_L=a$ and for every a there exists x such that $a+x=0_L$ and for all a, b, c holds $(a+b)+c=a+(b+c)$ and for all a, b holds $a+b=b+a$ and $0_L \neq \mathbf{1}_L$ and for every a holds $a\cdot\mathbf{1}_L=a$ and for every a holds $\mathbf{1}_L\cdot a=a$ and for all a, b such that $a\neq 0_L$ there exists x such that $a\cdot x=b$ and for all a, b such that $a\neq 0_L$ there exists x such that $x\cdot a=b$ and for all a, x, y such that $a\neq 0_L$ holds if $a\cdot x=a\cdot y$, then $x=y$ and for all a, x, y such that $a\neq 0_L$ holds if $x\cdot a=y\cdot a$, then $x=y$ and for every a holds $a\cdot 0_L=0_L$ and for every a holds $0_L\cdot a=0_L$ and for all a, b, c holds $(b+c)\cdot a=b\cdot a+c\cdot a$.

We use the following convention: G denotes a left distributive double loop and a, b, x, y denote elements of G.

We now state three propositions:

(21)[5] $(-b)\cdot a=-b\cdot a$.

(23)[6] For every Abelian left distributive double loop G holds $(-\mathbf{1}_G)\cdot -\mathbf{1}_G=\mathbf{1}_G$.

(24) $(x-y)\cdot a=x\cdot a-y\cdot a$.

A double sided quasi-field is an Abelian add-associative distributive non degenerated double loop.

In the sequel a, b, c, x, y denote elements of L.
Next we state the proposition

(26)[7] L is a double sided quasi-field if and only if the following conditions are satisfied:

for every a holds $a+0_L=a$ and for every a there exists x such that $a+x=0_L$ and for all a, b, c holds $(a+b)+c=a+(b+c)$ and for all a, b holds $a+b=b+a$ and $0_L \neq \mathbf{1}_L$ and for every a holds $a\cdot\mathbf{1}_L=a$ and for every a holds $\mathbf{1}_L\cdot a=a$ and for all a, b such that $a\neq 0_L$ there exists x such that $a\cdot x=b$ and for all a, b such that $a\neq 0_L$ there exists x such that $x\cdot a=b$ and for all a, x, y such that $a\neq 0_L$ holds if $a\cdot x=a\cdot y$, then $x=y$ and for all a, x, y such that $a\neq 0_L$ holds if $x\cdot a=y\cdot a$, then $x=y$ and for every a holds $a\cdot 0_L=0_L$ and for every a holds $0_L\cdot a=0_L$ and for all a, b, c holds $a\cdot(b+c)=a\cdot b+a\cdot c$ and for all a, b, c holds $(b+c)\cdot a=b\cdot a+c\cdot a$.

[2] The propositions (1)–(11) have been removed.
[3] The proposition (13) has been removed.
[4] The proposition (18) has been removed.
[5] The proposition (20) has been removed.
[6] The proposition (22) has been removed.
[7] The proposition (25) has been removed.

A skew field is an associative double sided quasi-field.
We now state the proposition

(32)[8] L is a skew field if and only if the following conditions are satisfied:

for every a holds $a+0_L = a$ and for every a there exists x such that $a+x = 0_L$ and for all a, b, c holds $(a+b)+c = a+(b+c)$ and for all a, b holds $a+b = b+a$ and $0_L \neq \mathbf{1}_L$ and for every a holds $a \cdot \mathbf{1}_L = a$ and for every a such that $a \neq 0_L$ there exists x such that $a \cdot x = \mathbf{1}_L$ and for every a holds $a \cdot 0_L = 0_L$ and for every a holds $0_L \cdot a = 0_L$ and for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and for all a, b, c holds $a \cdot (b+c) = a \cdot b + a \cdot c$ and for all a, b, c holds $(b+c) \cdot a = b \cdot a + c \cdot a$.

A field is a commutative skew field.
One can prove the following proposition

(34)[9] L is a field if and only if the following conditions are satisfied:

for every a holds $a+0_L = a$ and for every a there exists x such that $a+x = 0_L$ and for all a, b, c holds $(a+b)+c = a+(b+c)$ and for all a, b holds $a+b = b+a$ and $0_L \neq \mathbf{1}_L$ and for every a holds $a \cdot \mathbf{1}_L = a$ and for every a such that $a \neq 0_L$ there exists x such that $a \cdot x = \mathbf{1}_L$ and for every a holds $a \cdot 0_L = 0_L$ and for all a, b, c holds $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ and for all a, b, c holds $a \cdot (b+c) = a \cdot b + a \cdot c$ and for all a, b holds $a \cdot b = b \cdot a$.

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[8] The propositions (27)–(31) have been removed.
[9] The proposition (33) has been removed.