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Technical Preliminaries to Algebraic Specifications

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The articles [14], [18], [19], [6], [17], [5], [20], [8], [9], [10], [15], [11], [7], [1], [16], [13], [2], [12], [4], and [3] provide the notation and terminology for this paper.

1. PRELIMINARIES

The following propositions are true:

(1) For all functions f, g, h such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \subseteq \mathrm{dom}\, h$ holds $f{+\cdot}g{+\cdot}h = g{+\cdot}f{+\cdot}h$.

(2) For all functions f, g, h such that $f \subseteq g$ and $\mathrm{rng}\, h \cap \mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ holds $g \cdot h = f \cdot h$.

(3) For all functions f, g, h such that $\mathrm{dom}\, f \subseteq \mathrm{rng}\, g$ and $\mathrm{dom}\, f$ misses $\mathrm{rng}\, h$ and $g^{\circ}\, \mathrm{dom}\, h$ misses $\mathrm{dom}\, f$ holds $f \cdot (g{+\cdot}h) = f \cdot g$.

(4) For all functions f_1, f_2, g_1, g_2 such that $f_1 \approx f_2$ and $g_1 \approx g_2$ holds $f_1 \cdot g_1 \approx f_2 \cdot g_2$.

(5) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \subseteq g$, then $f^* \subseteq g^*$.

(6) Let X_1, Y_1, X_2, Y_2 be non empty sets, f be a function from X_1 into X_2, and g be a function from Y_1 into Y_2. If $f \approx g$, then $f^* \approx g^*$.

Let X be a set and let f be a function. The functor X -indexing f yields a many sorted set indexed by X and is defined by:

(Def. 1) X -indexing $f = \mathrm{id}_X{+\cdot}f{\restriction}X$.

One can prove the following propositions:

(7) For every set X and for every function f holds $\mathrm{rng}(X\text{ -indexing}\, f) = (X \setminus \mathrm{dom}\, f) \cup f^{\circ}X$.

(8) For every non empty set X and for every function f and for every element x of X holds $(X\text{ -indexing}\, f)(x) = (\mathrm{id}_X{+\cdot}f)(x)$.

(9) For all sets X, x and for every function f such that $x \in X$ holds if $x \in \mathrm{dom}\, f$, then $(X\text{ -indexing}\, f)(x) = f(x)$ and if $x \notin \mathrm{dom}\, f$, then $(X\text{ -indexing}\, f)(x) = x$.

(10) For every set X and for every function f such that $\mathrm{dom}\, f = X$ holds X -indexing $f = f$.

(11) For every set X and for every function f holds $X\text{ -indexing}(X\text{ -indexing}\, f) = X\text{ -indexing}\, f$.

(12) For every set X and for every function f holds $X\text{ -indexing}(\mathrm{id}_X{+\cdot}f) = X\text{ -indexing}\, f$.

(13) For every set X and for every function f such that $f \subseteq \mathrm{id}_X$ holds X-indexing $f = \mathrm{id}_X$.

(14) For every set X holds X-indexing $\emptyset = \mathrm{id}_X$.

(15) For every set X and for every function f holds X-indexing $f \restriction X = X$-indexing f.

(16) For every set X and for every function f such that $X \subseteq \mathrm{dom} f$ holds X-indexing $f = f \restriction X$.

(17) For every function f holds $\emptyset$-indexing $f = \emptyset$.

(18) For all sets X, Y and for every function f such that $X \subseteq Y$ holds $(Y$-indexing $f) \restriction X =$ X-indexing f.

(19) For all sets X, Y and for every function f holds $(X \cup Y)$-indexing $f = (X$-indexing $f) +\cdot (Y$-indexing $f)$.

(20) For all sets X, Y and for every function f holds X-indexing $f \approx Y$-indexing f.

(21) For all sets X, Y and for every function f holds $(X \cup Y)$-indexing $f = (X$-indexing $f) \cup$ $(Y$-indexing $f)$.

(22) For every non empty set X and for all functions f, g such that $\mathrm{rng}\, g \subseteq X$ holds $(X$-indexing $f) \cdot g = (\mathrm{id}_X +\cdot f) \cdot g$.

(23) For all functions f, g such that $\mathrm{dom} f$ misses $\mathrm{dom}\, g$ and $\mathrm{rng}\, g$ misses $\mathrm{dom} f$ and for every set X holds $f \cdot (X$-indexing $g) = f \restriction X$.

Let f be a function. A function is called a rng-retraction of f if:

(Def. 2) $\mathrm{dom}\,\mathrm{it} = \mathrm{rng} f$ and $f \cdot \mathrm{it} = \mathrm{id}_{\mathrm{rng} f}$.

One can prove the following propositions:

(24) For every function f and for every rng-retraction g of f holds $\mathrm{rng}\, g \subseteq \mathrm{dom} f$.

(25) Let f be a function, g be a rng-retraction of f, and x be a set. If $x \in \mathrm{rng} f$, then $g(x) \in \mathrm{dom} f$ and $f(g(x)) = x$.

(26) For every function f such that f is one-to-one holds f^{-1} is a rng-retraction of f.

(27) For every function f such that f is one-to-one and for every rng-retraction g of f holds $g = f^{-1}$.

(28) Let f_1, f_2 be functions. Suppose $f_1 \approx f_2$. Let g_1 be a rng-retraction of f_1 and g_2 be a rng-retraction of f_2. Then $g_1 +\cdot g_2$ is a rng-retraction of $f_1 +\cdot f_2$.

(29) Let f_1, f_2 be functions. Suppose $f_1 \subseteq f_2$. Let g_1 be a rng-retraction of f_1. Then there exists a rng-retraction g_2 of f_2 such that $g_1 \subseteq g_2$.

2. Replacement in Signature

Let S be a non empty non void many sorted signature and let f, g be functions. We say that f and g form a replacement in S if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let o_1, o_2 be operation symbols of S. Suppose $(\mathrm{id}_{\text{the operation symbols of } S} +\cdot g)(o_1) = (\mathrm{id}_{\text{the operation symbols of } S} +\cdot g)(o_2)$. Then

- (i) $(\mathrm{id}_{\text{the carrier of } S} +\cdot f) \cdot \mathrm{Arity}(o_1) = (\mathrm{id}_{\text{the carrier of } S} +\cdot f) \cdot \mathrm{Arity}(o_2)$, and
- (ii) $(\mathrm{id}_{\text{the carrier of } S} +\cdot f)(\text{the result sort of } o_1) = (\mathrm{id}_{\text{the carrier of } S} +\cdot f)(\text{the result sort of } o_2)$.

Next we state two propositions:

(30) Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if for all operation symbols o_1, o_2 of S such that ((the operation symbols of S) -indexing $g)(o_1) =$ ((the operation symbols of S) -indexing $g)(o_2)$ holds ((the carrier of S) -indexing $f) \cdot \mathrm{Arity}(o_1) =$ ((the carrier of S) -indexing $f) \cdot \mathrm{Arity}(o_2)$ and ((the carrier of S) -indexing $f)$(the result sort of $o_1) =$ ((the carrier of S) -indexing $f)$(the result sort of o_2).

(31) Let S be a non empty non void many sorted signature and f, g be functions. Then f and g form a replacement in S if and only if (the carrier of S) -indexing f and (the operation symbols of S) -indexing g form a replacement in S.

In the sequel S, S' are non void signatures and f, g are functions.
One can prove the following four propositions:

(32) If f and g form morphism between S and S', then f and g form a replacement in S.

(33) f and $\emptyset$ form a replacement in S.

(34) If g is one-to-one and (the operation symbols of $S) \cap \mathrm{rng}\, g \subseteq \mathrm{dom}\, g$, then f and g form a replacement in S.

(35) If g is one-to-one and $\mathrm{rng}\, g$ misses the operation symbols of S, then f and g form a replacement in S.

Let X be a set, let Y be a non empty set, let a be a function from Y into X^*, and let r be a function from Y into X. Note that $\langle X, Y, a, r \rangle$ is non void.

Let S be a non empty non void many sorted signature and let f, g be functions. Let us assume that f and g form a replacement in S. The functor S with-replacement(f, g) yields a strict non empty non void many sorted signature and is defined by the conditions (Def. 4).

(Def. 4)(i) (The carrier of S) -indexing f and (the operation symbols of S) -indexing g form morphism between S and S with-replacement(f, g),

(ii) the carrier of S with-replacement$(f, g) = \mathrm{rng}$((the carrier of S) -indexing f), and

(iii) the operation symbols of S with-replacement$(f, g) = \mathrm{rng}$((the operation symbols of S) -indexing g).

We now state a number of propositions:

(36) Let S_1, S_2 be non void signatures, f be a function from the carrier of S_1 into the carrier of S_2, and g be a function. Suppose f and g form morphism between S_1 and S_2. Then $f^* \cdot$ the arity of $S_1 =$ (the arity of $S_2) \cdot g$.

(37) Suppose f and g form a replacement in S. Then (the carrier of S) -indexing f is a function from the carrier of S into the carrier of S with-replacement(f, g).

(38) Suppose f and g form a replacement in S. Let f' be a function from the carrier of S into the carrier of S with-replacement(f, g). Suppose $f' =$ (the carrier of S) -indexing f. Let g' be a rng-retraction of (the operation symbols of S) -indexing g. Then the arity of S with-replacement$(f, g) = f'^* \cdot$ the arity of $S \cdot g'$.

(39) Suppose f and g form a replacement in S. Let g' be a rng-retraction of (the operation symbols of S) -indexing g. Then the result sort of S with-replacement$(f, g) =$ ((the carrier of S) -indexing $f) \cdot$ the result sort of $S \cdot g'$.

(40) If f and g form morphism between S and S', then S with-replacement(f, g) is a subsignature of S'.

(41) f and g form a replacement in S if and only if (the carrier of S) -indexing f and (the operation symbols of S) -indexing g form morphism between S and S with-replacement(f, g).

(42) Suppose dom $f \subseteq$ the carrier of S and dom $g \subseteq$ the operation symbols of S and f and g form a replacement in S. Then $\mathrm{id}_{\text{the carrier of } S} {+\cdot} f$ and $\mathrm{id}_{\text{the operation symbols of } S} {+\cdot} g$ form morphism between S and S with-replacement(f,g).

(43) Suppose dom $f =$ the carrier of S and dom $g =$ the operation symbols of S and f and g form a replacement in S. Then f and g form morphism between S and S with-replacement(f,g).

(44) If f and g form a replacement in S, then S with-replacement((the carrier of S) -indexing $f,g) = S$ with-replacement(f,g).

(45) If f and g form a replacement in S, then S with-replacement$(f,$(the operation symbols of S) -indexing $g) = S$ with-replacement(f,g).

3. Signature Extensions

Let S be a signature. A signature is called an extension of S if:

(Def. 5) S is a subsignature of it.

The following propositions are true:

(47)[1] Every signature S is an extension of S.

(48) For every signature S_1 and for every extension S_2 of S_1 holds every extension of S_2 is an extension of S_1.

(49) For all non empty signatures S_1, S_2 such that $S_1 \approx S_2$ holds $S_1 {+\cdot} S_2$ is an extension of S_1.

(50) For all non empty signatures S_1, S_2 holds $S_1 {+\cdot} S_2$ is an extension of S_2.

(51) Let S_1, S_2, S be non empty many sorted signatures and f_1, g_1, f_2, g_2 be functions. Suppose $f_1 \approx f_2$ and f_1 and g_1 form morphism between S_1 and S and f_2 and g_2 form morphism between S_2 and S. Then $f_1 {+\cdot} f_2$ and $g_1 {+\cdot} g_2$ form morphism between $S_1 {+\cdot} S_2$ and S.

(52) Let S_1, S_2, E be non empty signatures. Then E is an extension of S_1 and an extension of S_2 if and only if $S_1 \approx S_2$ and E is an extension of $S_1 {+\cdot} S_2$.

Let S be a non empty signature. One can verify that every extension of S is non empty.

Let S be a non void signature. Observe that every extension of S is non void.

Next we state the proposition

(53) For all signatures S, T such that S is empty holds T is an extension of S.

Let S be a signature. Note that there exists an extension of S which is non empty, non void, and strict.

We now state three propositions:

(54) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then f and g form a replacement in S.

(55) Let S be a non void signature and E be an extension of S. Suppose f and g form a replacement in E. Then E with-replacement(f,g) is an extension of S with-replacement(f,g).

(56) Let S_1, S_2 be non void signatures. Suppose $S_1 \approx S_2$. Let f, g be functions. If f and g form a replacement in $S_1 {+\cdot} S_2$, then $(S_1 {+\cdot} S_2)$ with-replacement$(f,g) = (S_1$ with-replacement$(f,g)) {+\cdot} (S_2$ with-replacement$(f,g))$.

[1] The proposition (46) has been removed.

4. ALGEBRAS

Algebra is defined by:

(Def. 6) There exists a non void signature S such that it is a feasible algebra over S.

Let S be a signature. An algebra is called an algebra of S if:

(Def. 7) There exists a non void extension E of S such that it is a feasible algebra over E.

Next we state a number of propositions:

(57) For every non void signature S holds every feasible algebra over S is an algebra of S.

(58) For every signature S and for every extension E of S holds every algebra of E is an algebra of S.

(59) Let S be a signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Then the carrier of $S \subseteq$ the carrier of E and the operation symbols of $S \subseteq$ the operation symbols of E.

(60) Let S be a non void signature, E be a non empty signature, and A be an algebra over E. Suppose A is an algebra of S. Let o be an operation symbol of S. Then (the characteristics of $A)(o)$ is a function from (the sorts of $A)^{\#}(\mathrm{Arity}(o))$ into (the sorts of $A)$(the result sort of o).

(61) Let S be a non empty signature, A be an algebra of S, and E be a non empty many sorted signature. If A is an algebra over E, then A is an algebra over $E + \cdot S$.

(62) Let S_1, S_2 be non empty signatures and A be an algebra over S_1. Suppose A is an algebra over S_2. Then the carrier of $S_1 =$ the carrier of S_2 and the operation symbols of $S_1 =$ the operation symbols of S_2.

(63) Let S be a non void signature and A be a non-empty disjoint valued algebra over S. Then the sorts of A are one-to-one.

(64) Let S be a non void signature, A be a disjoint valued algebra over S, and C_1, C_2 be components of the sorts of A. Then $C_1 = C_2$ or C_1 misses C_2.

(65) Let S, S' be non void signatures and A be a non-empty disjoint valued algebra over S. Suppose A is an algebra over S'. Then the many sorted signature of $S =$ the many sorted signature of S'.

(66) Let S' be a non void signature and A be a non-empty disjoint valued algebra over S. If A is an algebra of S', then S is an extension of S'.

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