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Directed Geometrical Bundles and Their Analytical Representation[1]

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Summary. We introduce the notion of weak directed geometrical bundle. We prove representation theorems for directed and weak directed geometrical bundles which establishes a one-to-one correspondence between such structures and appropriate 2-divisible abelian groups. To this aim we construct over arbitrary weak directed geometrical bundle a group defined entirely in terms of geometrical notions – the group of (abstract) "free vectors".

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The articles [9], [5], [11], [1], [8], [7], [3], [4], [2], [12], [6], and [10] provide the notation and terminology for this paper.

Let I_1 be a non empty affine structure. We say that I_1 is weak affine vector space-like if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For all elements a, b, c of I_1 such that $a,b \upuparrows c,c$ holds $a=b$,

(ii) for all elements a, b, c, d, p, q of I_1 such that $a,b \upuparrows p,q$ and $c,d \upuparrows p,q$ holds $a,b \upuparrows c,d$,

(iii) for all elements a, b, c of I_1 there exists an element d of I_1 such that $a,b \upuparrows c,d$,

(iv) for all elements a, b, c, a', b', c' of I_1 such that $a,b \upuparrows a',b'$ and $a,c \upuparrows a',c'$ holds $b,c \upuparrows b',c'$,

(v) for all elements a, c of I_1 there exists an element b of I_1 such that $a,b \upuparrows b,c$, and

(vi) for all elements a, b, c, d of I_1 such that $a,b \upuparrows c,d$ holds $a,c \upuparrows b,d$.

Let us observe that there exists a non empty affine structure which is strict, non trivial, and weak affine vector space-like.

A weak affine vector space is a non trivial weak affine vector space-like non empty affine structure.

Let us note that every non empty affine structure which is space of free vectors-like is also weak affine vector space-like.

We follow the rules: A_1 is a weak affine vector space and a, b, c, d, f, a', b', c', d', f', p, q, r, o are elements of A_1.

Next we state a number of propositions:

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(2)[1] $a,b \downharpoonleft\!\upharpoonright a,b$.

(3) $a,a \downharpoonleft\!\upharpoonright a,a$.

(4) If $a,b \downharpoonleft\!\upharpoonright c,d$, then $c,d \downharpoonleft\!\upharpoonright a,b$.

(5) If $a,b \downharpoonleft\!\upharpoonright a,c$, then $b = c$.

(6) If $a,b \downharpoonleft\!\upharpoonright c,d$ and $a,b \downharpoonleft\!\upharpoonright c,d'$, then $d = d'$.

(7) For all a, b holds $a,a \downharpoonleft\!\upharpoonright b,b$.

(8) If $a,b \downharpoonleft\!\upharpoonright c,d$, then $b,a \downharpoonleft\!\upharpoonright d,c$.

(9) If $a,b \downharpoonleft\!\upharpoonright c,d$ and $a,c \downharpoonleft\!\upharpoonright b',d$, then $b = b'$.

(10) If $b,c \downharpoonleft\!\upharpoonright b',c'$ and $a,d \downharpoonleft\!\upharpoonright b,c$ and $a,d' \downharpoonleft\!\upharpoonright b',c'$, then $d = d'$.

(11) If $a,b \downharpoonleft\!\upharpoonright a',b'$ and $c,d \downharpoonleft\!\upharpoonright b,a$ and $c,d' \downharpoonleft\!\upharpoonright b',a'$, then $d = d'$.

(12) If $a,b \downharpoonleft\!\upharpoonright a',b'$ and $c,d \downharpoonleft\!\upharpoonright c',d'$ and $b,f \downharpoonleft\!\upharpoonright c,d$ and $b',f' \downharpoonleft\!\upharpoonright c',d'$, then $a,f \downharpoonleft\!\upharpoonright a',f'$.

(13) If $a,b \downharpoonleft\!\upharpoonright a',b'$ and $a,c \downharpoonleft\!\upharpoonright c',b'$, then $b,c \downharpoonleft\!\upharpoonright c',a'$.

Let us consider A_1 and let us consider a, b. We say that a, b are in a maximal distance if and only if:

(Def. 2) $a,b \downharpoonleft\!\upharpoonright b,a$ and $a \neq b$.

Let us notice that the predicate a, b are in a maximal distance is irreflexive and symmetric.

The following three propositions are true:

(16)[2] There exist a, b such that $a \neq b$ and a, b are not in a maximal distance.

(18)[3] Suppose a, b are in a maximal distance and a, c are in a maximal distance. Then $b = c$ or b, c are in a maximal distance.

(19) If a, b are in a maximal distance and $a,b \downharpoonleft\!\upharpoonright c,d$, then c, d are in a maximal distance.

Let us consider A_1 and let us consider a, b, c. We say that b is a midpoint of a, c if and only if:

(Def. 3) $a,b \downharpoonleft\!\upharpoonright b,c$.

Next we state a number of propositions:

(21)[4] If b is a midpoint of a, c, then b is a midpoint of c, a.

(22) b is a midpoint of a, b iff $a = b$.

(23) b is a midpoint of a, a iff $a = b$ or a, b are in a maximal distance.

(24) There exists b such that b is a midpoint of a, c.

(25) Suppose b is a midpoint of a, c and b' is a midpoint of a, c. Then $b = b'$ or b, b' are in a maximal distance.

(26) There exists c such that b is a midpoint of a, c.

(27) If b is a midpoint of a, c and b is a midpoint of a, c', then $c = c'$.

(28) If b is a midpoint of a, c and b, b' are in a maximal distance, then b' is a midpoint of a, c.

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[1] The proposition (1) has been removed.
[2] The propositions (14) and (15) have been removed.
[3] The proposition (17) has been removed.
[4] The proposition (20) has been removed.

(29) Suppose b is a midpoint of a, c and b' is a midpoint of a, c' and b, b' are in a maximal distance. Then $c = c'$.

(30) If p is a midpoint of a, a' and p is a midpoint of b, b', then $a,b \upharpoonleft\upharpoonright b',a'$.

(31) Suppose p is a midpoint of a, a' and q is a midpoint of b, b' and p, q are in a maximal distance. Then $a,b \upharpoonleft\upharpoonright b',a'$.

Let us consider A_1 and let us consider a, b. The functor $\mathrm{PSym}(a,b)$ yielding an element of A_1 is defined by:

(Def. 4) a is a midpoint of b, $\mathrm{PSym}(a,b)$.

One can prove the following propositions:

(33)[5] $\mathrm{PSym}(p,a) = b$ iff $a,p \upharpoonleft\upharpoonright p,b$.

(35)[6] $\mathrm{PSym}(p,a) = a$ iff $a = p$ or a, p are in a maximal distance.

(36) $\mathrm{PSym}(p,\mathrm{PSym}(p,a)) = a$.

(37) If $\mathrm{PSym}(p,a) = \mathrm{PSym}(p,b)$, then $a = b$.

(38) There exists a such that $\mathrm{PSym}(p,a) = b$.

(39) $a,b \upharpoonleft\upharpoonright \mathrm{PSym}(p,b),\mathrm{PSym}(p,a)$.

(40) $a,b \upharpoonleft\upharpoonright c,d$ iff $\mathrm{PSym}(p,a),\mathrm{PSym}(p,b) \upharpoonleft\upharpoonright \mathrm{PSym}(p,c),\mathrm{PSym}(p,d)$.

(41) a, b are in a maximal distance iff $\mathrm{PSym}(p,a)$, $\mathrm{PSym}(p,b)$ are in a maximal distance.

(42) b is a midpoint of a, c iff $\mathrm{PSym}(p,b)$ is a midpoint of $\mathrm{PSym}(p,a)$, $\mathrm{PSym}(p,c)$.

(43) $\mathrm{PSym}(p,a) = \mathrm{PSym}(q,a)$ iff $p = q$ or p, q are in a maximal distance.

(44) $\mathrm{PSym}(q,\mathrm{PSym}(p,\mathrm{PSym}(q,a))) = \mathrm{PSym}(\mathrm{PSym}(q,p),a)$.

(45) $\mathrm{PSym}(p,\mathrm{PSym}(q,a)) = \mathrm{PSym}(q,\mathrm{PSym}(p,a))$ if and only if one of the following conditions is satisfied:

(i) $p = q$, or

(ii) p, q are in a maximal distance, or

(iii) q, $\mathrm{PSym}(p,q)$ are in a maximal distance.

(46) $\mathrm{PSym}(p,\mathrm{PSym}(q,\mathrm{PSym}(r,a))) = \mathrm{PSym}(r,\mathrm{PSym}(q,\mathrm{PSym}(p,a)))$.

(47) There exists d such that $\mathrm{PSym}(a,\mathrm{PSym}(b,\mathrm{PSym}(c,p))) = \mathrm{PSym}(d,p)$.

(48) There exists c such that $\mathrm{PSym}(a,\mathrm{PSym}(c,p)) = \mathrm{PSym}(c,\mathrm{PSym}(b,p))$.

Let us consider A_1, o and let us consider a, b. The functor $\mathrm{Padd}(o,a,b)$ yielding an element of A_1 is defined as follows:

(Def. 5) $o,a \upharpoonleft\upharpoonright b,\mathrm{Padd}(o,a,b)$.

Let us consider A_1, o and let us consider a. We introduce $\mathrm{Pcom}(o,a)$ as a synonym of $\mathrm{PSym}(o,a)$.

Let us consider A_1, o. The functor $\mathrm{Padd}\,o$ yields a binary operation on the carrier of A_1 and is defined by:

(Def. 7)[7] For all a, b holds $(\mathrm{Padd}\,o)(a, b) = \mathrm{Padd}(o,a,b)$.

[5] The proposition (32) has been removed.
[6] The proposition (34) has been removed.
[7] The definition (Def. 6) has been removed.

Let us consider A_1, o. The functor Pcom$o$ yields a unary operation on the carrier of A_1 and is defined as follows:

(Def. 8) For every a holds $(\mathrm{Pcom}\, o)(a) = \mathrm{Pcom}(o,a)$.

Let us consider A_1, o. The functor GroupVect(A_1,o) yields a strict loop structure and is defined by:

(Def. 9) $\mathrm{GroupVect}(A_1,o) = \langle$the carrier of A_1, Padd$o,o\rangle$.

Let us consider A_1, o. One can verify that GroupVect(A_1,o) is non empty.

Next we state two propositions:

(55)[8] The carrier of GroupVect$(A_1,o) =$ the carrier of A_1 and the addition of GroupVect$(A_1,o) =$ Padd$o$ and the zero of GroupVect$(A_1,o) = o$.

(57)[9] For all elements a, b of GroupVect(A_1,o) and for all elements a', b' of A_1 such that $a = a'$ and $b = b'$ holds $a+b = (\mathrm{Padd}\, o)(a', b')$.

Let us consider A_1, o. Note that GroupVect(A_1,o) is Abelian, add-associative, right zeroed, and right complementable.

Next we state two propositions:

(58) For every element a of GroupVect(A_1,o) and for every element a' of A_1 such that $a = a'$ holds $-a = (\mathrm{Pcom}\, o)(a')$.

(59) $0_{\mathrm{GroupVect}(A_1,o)} = o$.

In the sequel a, b denote elements of GroupVect(A_1,o).

Next we state the proposition

(66)[10] For every a there exists b such that $b+b = a$.

Let us consider A_1, o. One can verify that GroupVect(A_1,o) is 2-divisible.

In the sequel A_1 denotes a space of free vectors and o denotes an element of A_1.

Next we state the proposition

(67) For every element a of GroupVect(A_1,o) such that $a + a = 0_{\mathrm{GroupVect}(A_1,o)}$ holds $a = 0_{\mathrm{GroupVect}(A_1,o)}$.

Let us consider A_1, o. One can check that GroupVect(A_1,o) is Fanoian.

Let us note that there exists a uniquely 2-divisible group which is strict and non trivial.

A proper uniquely two divisible group is a non trivial uniquely 2-divisible group.

Next we state the proposition

(69)[11] GroupVect(A_1,o) is a proper uniquely two divisible group.

Let us consider A_1, o. Note that GroupVect(A_1,o) is non trivial.

Next we state the proposition

(70) For every proper uniquely two divisible group A_2 holds Vectors(A_2) is a space of free vectors.

Let A_2 be a proper uniquely two divisible group. Observe that Vectors(A_2) is space of free vectors-like and non trivial.

One can prove the following two propositions:

[8] The propositions (49)–(54) have been removed.
[9] The proposition (56) has been removed.
[10] The propositions (60)–(65) have been removed.
[11] The proposition (68) has been removed.

(71) For every strict space A_1 of free vectors and for every element o of A_1 holds $A_1 = \mathrm{Vectors}(\mathrm{GroupVect}(A_1, o))$.

(72) Let A_3 be a strict affine structure. Then A_3 is a space of free vectors if and only if there exists a proper uniquely two divisible group A_2 such that $A_3 = \mathrm{Vectors}(A_2)$.

Let X, Y be non empty loop structures and let f be a function from the carrier of X into the carrier of Y. We say that f is an isomorphism of X and Y if and only if the conditions (Def. 10) are satisfied.

(Def. 10)(i) f is one-to-one,

(ii) $\mathrm{rng}\, f =$ the carrier of Y, and

(iii) for all elements a, b of X holds $f(a+b) = f(a) + f(b)$ and $f(0_X) = 0_Y$ and $f(-a) = -f(a)$.

Let X, Y be non empty loop structures. We say that X, Y are isomorph if and only if:

(Def. 11) There exists a function from the carrier of X into the carrier of Y which is an isomorphism of X and Y.

In the sequel A_2 denotes a proper uniquely two divisible group and f denotes a function from the carrier of A_2 into the carrier of A_2.

Next we state four propositions:

(75)[12] Let o' be an element of A_2 and o be an element of $\mathrm{Vectors}(A_2)$. Suppose for every element x of A_2 holds $f(x) = o' + x$ and $o = o'$. Let a, b be elements of A_2. Then $f(a+b) = (\mathrm{Padd}\, o)(f(a), f(b))$ and $f(0_{(A_2)}) = 0_{\mathrm{GroupVect}(\mathrm{Vectors}(A_2),o)}$ and $f(-a) = (\mathrm{Pcom}\, o)(f(a))$.

(76) For every element o' of A_2 such that for every element b of A_2 holds $f(b) = o' + b$ holds f is one-to-one.

(77) Let o' be an element of A_2 and o be an element of $\mathrm{Vectors}(A_2)$. Suppose that for every element b of A_2 holds $f(b) = o' + b$. Then $\mathrm{rng}\, f =$ the carrier of $\mathrm{GroupVect}(\mathrm{Vectors}(A_2), o)$.

(78) Let A_2 be a proper uniquely two divisible group, o' be an element of A_2, and o be an element of $\mathrm{Vectors}(A_2)$. If $o = o'$, then A_2, $\mathrm{GroupVect}(\mathrm{Vectors}(A_2), o)$ are isomorph.

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