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Abian's Fixed Point Theorem[1]

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Summary. A. Abian [1] proved the following theorem:

Let f be a mapping from a finite set D. Then f has a fixed point if and only if D is not a union of three mutually disjoint sets A, B and C such that

$$A \cap f[A] = B \cap f[B] = C \cap f[C] = \emptyset.$$

(The range of f is not necessarily the subset of its domain). The proof of the sufficiency is by induction on the number of elements of D. A. Mąkowski and K. Wiśniewski [10] have shown that the assumption of finiteness is superfluous. They proved their version of the theorem for f being a function from D into D. In the proof, the required partition was constructed and the construction used the axiom of choice. Their main point was to demonstrate that the use of this axiom in the proof is essential. We have proved in Mizar the generalized version of Abian's theorem, i.e. without assuming finiteness of D. We have simplified the proof from [10] which uses well-ordering principle and transfinite ordinals—our proof does not use these notions but otherwise is based on their idea (we employ choice functions).

MML Identifier: `ABIAN`.

WWW: `http://mizar.org/JFM/Vol9/abian.html`

The articles [15], [7], [19], [3], [17], [16], [8], [11], [9], [20], [5], [6], [14], [18], [2], [12], [4], and [13] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: x, y, E, E_1, E_2, E_3 are sets, s_1 is a family of subsets of E, f is a function from E into E, and k, l, n are natural numbers.

Let i be a number. We say that i is even if and only if:

(Def. 1) There exists an integer j such that $i = 2 \cdot j$.

We introduce i is odd as an antonym of i is even.

Let n be a natural number. Let us observe that n is even if and only if:

(Def. 2) There exists k such that $n = 2 \cdot k$.

We introduce n is odd as an antonym of n is even.

One can verify the following observations:

* there exists a natural number which is even,
* there exists a natural number which is odd,
* there exists an integer which is even, and
* there exists an integer which is odd.

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The following proposition is true

(1) For every integer i holds i is odd iff there exists an integer j such that $i = 2 \cdot j + 1$.

Let i be an integer. Note that $2 \cdot i$ is even.
Let i be an even integer. Observe that $i+1$ is odd.
Let i be an odd integer. Note that $i+1$ is even.
Let i be an even integer. One can verify that $i-1$ is odd.
Let i be an odd integer. Observe that $i-1$ is even.
Let i be an even integer and let j be an integer. Observe that $i \cdot j$ is even and $j \cdot i$ is even.
Let i, j be odd integers. One can verify that $i \cdot j$ is odd.
Let i, j be even integers. Observe that $i+j$ is even.
Let i be an even integer and let j be an odd integer. One can check that $i+j$ is odd and $j+i$ is odd.
Let i, j be odd integers. Observe that $i+j$ is even.
Let i be an even integer and let j be an odd integer. One can check that $i-j$ is odd and $j-i$ is odd.
Let i, j be odd integers. Observe that $i-j$ is even.
Let us consider E, f, n. Then f^n is a function from E into E.
Let A be a set and let B be a set with a non-empty element. One can verify that there exists a function from A into B which is non-empty.
Let A be a non empty set, let B be a set with a non-empty element, let f be a non-empty function from A into B, and let a be an element of A. Note that $f(a)$ is non empty.
Let X be a non empty set. Note that 2^X has a non-empty element.
One can prove the following two propositions:

(2) For every non empty subset S of $\mathbb{N}$ such that $0 \in S$ holds $\min S = 0$.

(3) For every non empty set E and for every function f from E into E and for every element x of E holds $f^0(x) = x$.

Let f be a function. We say that f has a fixpoint if and only if:

(Def. 3) There exists x which is a fixpoint of f.

We introduce f has no fixpoint as an antonym of f has a fixpoint.
Let X be a set and let x be an element of X. We say that x is covering if and only if:

(Def. 4) $\bigcup x = \bigcup \bigcup X$.

One can prove the following proposition

(4) s_1 is covering iff $\bigcup s_1 = E$.

Let us consider E. One can verify that there exists a family of subsets of E which is non empty, finite, and covering.
One can prove the following proposition

(5) Let E be a set, f be a function from E into E, and s_1 be a non empty covering family of subsets of E such that for every element X of s_1 holds X misses $f^\circ X$. Then f has no fixpoint.

Let us consider E, f. The functor $f_\equiv$ yielding an equivalence relation of E is defined by:

(Def. 5) For all x, y such that $x \in E$ and $y \in E$ holds $\langle x, y \rangle \in f_\equiv$ iff there exist k, l such that $f^k(x) = f^l(y)$.

The following three propositions are true:

(6) Let E be a non empty set, f be a function from E into E, c be an element of $\mathrm{Classes}(f_\equiv)$, and e be an element of c. Then $f(e) \in c$.

(7) Let E be a non empty set, f be a function from E into E, c be an element of $\mathrm{Classes}(f_{\equiv})$, e be an element of c, and given n. Then $f^n(e) \in c$.

(8) Let E be a non empty set and f be a function from E into E. Suppose f has no fixpoint. Then there exist E_1, E_2, E_3 such that $E_1 \cup E_2 \cup E_3 = E$ and $f^\circ E_1$ misses E_1 and $f^\circ E_2$ misses E_2 and $f^\circ E_3$ misses E_3.

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