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On Semilattice Structure of Mizar Types

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Summary. The aim of this paper is to develop a formal theory of Mizar types. The presented theory is an approach to the structure of Mizar types as a sup-semilattice with widening (subtyping) relation as the order. It is an abstraction from the existing implementation of the Mizar verifier and formalization of the ideas from [9].

MML Identifier: ABCMIZ_0.

WWW: http://mizar.org/JFM/Vol15/abcmiz_0.html

The articles [18], [13], [22], [24], [21], [23], [3], [19], [1], [11], [12], [16], [10], [14], [15], [4], [2], [20], [5], [6], [7], [8], and [17] provide the notation and terminology for this paper.

1. Semilattice of Widening

Let us note that every non empty relational structure which is trivial and reflexive is also complete.

Let T be a relational structure. A type of T is an element of T.

Let T be a relational structure. We say that T is Noetherian if and only if:

(Def. 1) The internal relation of T is reversely well founded.

Let us observe that every non empty relational structure which is trivial is also Noetherian.

Let T be a non empty relational structure. Let us observe that T is Noetherian if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let A be a non empty subset of T. Then there exists an element a of T such that $a \in A$ and for every element b of T such that $b \in A$ holds $a \not< b$.

Let T be a poset. We say that T is Mizar-widening-like if and only if:

(Def. 3) T is a sup-semilattice and Noetherian.

One can verify that every poset which is Mizar-widening-like is also Noetherian and upper-bounded and has l.u.b.'s.

One can verify that every sup-semilattice which is Noetherian is also Mizar-widening-like.

One can verify that there exists a complete sup-semilattice which is Mizar-widening-like.

Let T be a Noetherian relational structure. One can check that the internal relation of T is reversely well founded.

The following proposition is true

(1) For every Noetherian sup-semilattice T and for every ideal I of T holds sup I exists in T and $\sup I \in I$.

2. ADJECTIVES

We introduce adjective structures which are systems

⟨ a set of adjectives, an operation non ⟩,

where the set of adjectives is a set and the operation non is a unary operation on the set of adjectives.

Let A be an adjective structure. We say that A is void if and only if:

(Def. 4) The set of adjectives of A is empty.

An adjective of A is an element of the set of adjectives of A.

Next we state the proposition

(2) Let A_1, A_2 be adjective structures. Suppose the set of adjectives of $A_1 =$ the set of adjectives of A_2. If A_1 is void, then A_2 is void.

Let A be an adjective structure and let a be an element of the set of adjectives of A. The functor non a yielding an adjective of A is defined as follows:

(Def. 5) $\mathrm{non}\, a =$ (the operation non of A)(a).

The following proposition is true

(3) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. Let a_1 be an adjective of A_1 and a_2 be an adjective of A_2. If $a_1 = a_2$, then $\mathrm{non}\, a_1 = \mathrm{non}\, a_2$.

Let A be an adjective structure. We say that A is involutive if and only if:

(Def. 6) For every adjective a of A holds $\mathrm{non}\,\mathrm{non}\, a = a$.

We say that A is without fixpoints if and only if:

(Def. 7) It is not true that there exists an adjective a of A such that $\mathrm{non}\, a = a$.

One can prove the following propositions:

(4) Let a_1, a_2 be sets. Suppose $a_1 \neq a_2$. Let A be an adjective structure. Suppose the set of adjectives of $A = \{a_1, a_2\}$ and (the operation non of A)$(a_1) = a_2$ and (the operation non of A)$(a_2) = a_1$. Then A is non void, involutive, and without fixpoints.

(5) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. If A_1 is involutive, then A_2 is involutive.

(6) Let A_1, A_2 be adjective structures. Suppose the adjective structure of $A_1 =$ the adjective structure of A_2. If A_1 is without fixpoints, then A_2 is without fixpoints.

Let us note that there exists a strict adjective structure which is non void, involutive, and without fixpoints.

Let A be a non void adjective structure. Note that the set of adjectives of A is non empty.

We introduce *TA*-structures which are extensions of relational structure and adjective structure and are systems

⟨ a carrier, a set of adjectives, an internal relation, an operation non, an adjective map ⟩,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, and the adjective map is a function from the carrier into Fin the set of adjectives.

Let X be a non empty set, let A be a set, let r be a binary relation on X, let n be a unary operation on A, and let a be a function from X into $\mathrm{Fin}\, A$. Observe that $\langle X, A, r, n, a \rangle$ is non empty.

Let X be a set, let A be a non empty set, let r be a binary relation on X, let n be a unary operation on A, and let a be a function from X into $\mathrm{Fin}\, A$. One can check that $\langle X, A, r, n, a \rangle$ is non void.

One can check that there exists a *TA*-structure which is trivial, reflexive, non empty, non void, involutive, without fixpoints, and strict.

Let T be a *TA*-structure and let t be an element of T. The functor adjs t yielding a subset of the set of adjectives of T is defined by:

(Def. 8) $\operatorname{adjs} t = (\text{the adjective map of } T)(t)$.

The following proposition is true

(7) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. Let t_1 be a type of T_1 and t_2 be a type of T_2. If $t_1 = t_2$, then $\operatorname{adjs} t_1 = \operatorname{adjs} t_2$.

Let T be a *TA*-structure. We say that T is consistent if and only if:

(Def. 9) For every type t of T and for every adjective a of T such that $a \in \operatorname{adjs} t$ holds $\operatorname{non} a \notin \operatorname{adjs} t$.

We now state the proposition

(8) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. If T_1 is consistent, then T_2 is consistent.

Let T be a non empty *TA*-structure. We say that T has structured adjectives if and only if:

(Def. 10) The adjective map of T is a join-preserving map from T into $(2_{\subseteq}^{\text{the set of adjectives of } T})^{\mathrm{op}}$.

Next we state the proposition

(9) Let T_1, T_2 be non empty *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. If T_1 has structured adjectives, then T_2 has structured adjectives.

Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s. Let us observe that T has structured adjectives if and only if:

(Def. 11) For all types t_1, t_2 of T holds $\operatorname{adjs}(t_1 \sqcup t_2) = \operatorname{adjs} t_1 \cap \operatorname{adjs} t_2$.

The following proposition is true

(10) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s. Suppose T has structured adjectives. Let t_1, t_2 be types of T. If $t_1 \leq t_2$, then $\operatorname{adjs} t_2 \subseteq \operatorname{adjs} t_1$.

Let T be a *TA*-structure and let a be an element of the set of adjectives of T. The functor $\operatorname{types} a$ yielding a subset of T is defined as follows:

(Def. 12) For every set x holds $x \in \operatorname{types} a$ iff there exists a type t of T such that $x = t$ and $a \in \operatorname{adjs} t$.

Let T be a non empty *TA*-structure and let A be a subset of the set of adjectives of T. The functor $\operatorname{types} A$ yields a subset of T and is defined as follows:

(Def. 13) For every type t of T holds $t \in \operatorname{types} A$ iff for every adjective a of T such that $a \in A$ holds $t \in \operatorname{types} a$.

The following propositions are true:

(11) Let T_1, T_2 be *TA*-structures. Suppose the *TA*-structure of $T_1 =$ the *TA*-structure of T_2. Let a_1 be an adjective of T_1 and a_2 be an adjective of T_2. If $a_1 = a_2$, then $\operatorname{types} a_1 = \operatorname{types} a_2$.

(12) For every non empty *TA*-structure T and for every adjective a of T holds $\operatorname{types} a = \{t; t$ ranges over types of T: $a \in \operatorname{adjs} t\}$.

(13) Let T be a *TA*-structure, t be a type of T, and a be an adjective of T. Then $a \in \operatorname{adjs} t$ if and only if $t \in \operatorname{types} a$.

(14) Let T be a non empty *TA*-structure, t be a type of T, and A be a subset of the set of adjectives of T. Then $A \subseteq \operatorname{adjs} t$ if and only if $t \in \operatorname{types} A$.

(15) For every non void *TA*-structure T and for every type t of T holds $\operatorname{adjs} t = \{a; a$ ranges over adjectives of T: $t \in \operatorname{types} a\}$.

(16) Let T be a non empty TA-structure and t be a type of T. Then $\mathrm{types}(\emptyset_{\text{the set of adjectives of } T}) =$ the carrier of T.

Let T be a TA-structure. We say that T has typed adjectives if and only if:

(Def. 14) For every adjective a of T holds $\mathrm{types}\, a \cup \mathrm{types}\, \mathrm{non}\, a$ is non empty.

The following proposition is true

(17) Let T_1, T_2 be TA-structures. Suppose the TA-structure of $T_1 =$ the TA-structure of T_2. If T_1 has typed adjectives, then T_2 has typed adjectives.

One can verify that there exists a complete upper-bounded non empty trivial reflexive transitive antisymmetric strict TA-structure which is non void, Mizar-widening-like, involutive, without fixpoints, and consistent and has structured adjectives and typed adjectives.

The following proposition is true

(18) For every consistent TA-structure T and for every adjective a of T holds $\mathrm{types}\, a$ misses $\mathrm{types}\, \mathrm{non}\, a$.

Let T be a reflexive transitive antisymmetric TA-structure with l.u.b.'s with structured adjectives and let a be an adjective of T. Observe that $\mathrm{types}\, a$ is lower and directed.

Let T be a reflexive transitive antisymmetric TA-structure with l.u.b.'s with structured adjectives and let A be a subset of the set of adjectives of T. Note that $\mathrm{types}\, A$ is lower and directed.

One can prove the following proposition

(19) Let T be reflexive antisymmetric transitive TA-structure with l.u.b.'s with structured adjectives and a be an adjective of T. Then $\mathrm{types}\, a$ is empty or $\mathrm{types}\, a$ is an ideal of T.

3. Applicability of Adjectives

Let T be a TA-structure, let t be an element of T, and let a be an adjective of T. We say that a is applicable to t if and only if:

(Def. 15) There exists a type t' of T such that $t' \in \mathrm{types}\, a$ and $t' \leq t$.

Let T be a TA-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. We say that A is applicable to t if and only if:

(Def. 16) There exists a type t' of T such that $A \subseteq \mathrm{adjs}\, t'$ and $t' \leq t$.

Next we state the proposition

(20) Let T be a reflexive transitive antisymmetric TA-structure with l.u.b.'s with structured adjectives, a be an adjective of T, and t be a type of T. If a is applicable to t, then $\mathrm{types}\, a \cap {\downarrow} t$ is an ideal of T.

Let T be a non empty reflexive transitive TA-structure, let t be an element of T, and let a be an adjective of T. The functor $a * t$ yields a type of T and is defined by:

(Def. 17) $a * t = \sup(\mathrm{types}\, a \cap {\downarrow} t)$.

The following propositions are true:

(21) Let T be a Noetherian reflexive transitive antisymmetric TA-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a * t \leq t$.

(22) Let T be a Noetherian reflexive transitive antisymmetric TA-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a \in \mathrm{adjs}(a * t)$.

(23) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If a is applicable to t, then $a * t \in \mathrm{types}\, a$.

(24) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, a be an adjective of T, and t' be a type of T. If $t' \leq t$ and $a \in \mathrm{adjs}\, t'$, then a is applicable to t and $t' \leq a * t$.

(25) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a be an adjective of T. If $a \in \mathrm{adjs}\, t$, then a is applicable to t and $a * t = t$.

(26) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and a, b be adjectives of T. Suppose a is applicable to t and b is applicable to $a * t$. Then b is applicable to t and a is applicable to $b * t$ and $a * (b * t) = b * (a * t)$.

(27) Let T be a reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, A be a subset of the set of adjectives of T, and t be a type of T. If A is applicable to t, then $\mathrm{types}\, A \cap {\downarrow} t$ is an ideal of T.

Let T be a non empty reflexive transitive *TA*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. The functor $A * t$ yielding a type of T is defined as follows:

(Def. 18) $A * t = \sup(\mathrm{types}\, A \cap {\downarrow} t)$.

One can prove the following proposition

(28) Let T be a non empty reflexive transitive antisymmetric *TA*-structure and t be a type of T. Then $\emptyset_{\text{the set of adjectives of } T} * t = t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let p be a finite sequence of elements of the set of adjectives of T. The functor $\mathrm{apply}(p,t)$ yields a finite sequence of elements of the carrier of T and is defined by the conditions (Def. 19).

(Def. 19)(i) $\mathrm{len}\,\mathrm{apply}(p,t) = \mathrm{len}\, p + 1$,

(ii) $(\mathrm{apply}(p,t))(1) = t$, and

(iii) for every natural number i and for every adjective a of T and for every type t of T such that $i \in \mathrm{dom}\, p$ and $a = p(i)$ and $t = (\mathrm{apply}(p,t))(i)$ holds $(\mathrm{apply}(p,t))(i+1) = a * t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let p be a finite sequence of elements of the set of adjectives of T. Observe that $\mathrm{apply}(p,t)$ is non empty.

The following two propositions are true:

(29) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\mathrm{apply}(\varepsilon_{(\text{the set of adjectives of } T)}, t) = \langle t \rangle$.

(30) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then $\mathrm{apply}(\langle a \rangle, t) = \langle t, a * t \rangle$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. The functor $v * t$ yields a type of T and is defined by:

(Def. 20) $v * t = (\mathrm{apply}(v,t))(\mathrm{len}\, v + 1)$.

Next we state several propositions:

(31) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)} * t = t$.

(32) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then $\langle a \rangle * t = a * t$.

(33) For all finite sequences p, q and for every natural number i such that $i \geq 1$ and $i < \operatorname{len} p$ holds $(p \mathbin{^{\$\frown}} q)(i) = p(i)$.

(34) Let p be a non empty finite sequence, q be a finite sequence, and i be a natural number. If $i < \operatorname{len} q$, then $(p \mathbin{^{\$\frown}} q)(\operatorname{len} p + i) = q(i+1)$.

(35) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $\operatorname{apply}(v_1 \frown v_2, t) = (\operatorname{apply}(v_1, t)) \mathbin{^{\$\frown}} \operatorname{apply}(v_2, v_1 * t)$.

(36) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, v_1, v_2 be finite sequences of elements of the set of adjectives of T, and i be a natural number. If $i \in \operatorname{dom} v_1$, then $(\operatorname{apply}(v_1 \frown v_2, t))(i) = (\operatorname{apply}(v_1, t))(i)$.

(37) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $(\operatorname{apply}(v_1 \frown v_2, t))(\operatorname{len} v_1 + 1) = v_1 * t$.

(38) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Then $v_2 * (v_1 * t) = (v_1 \frown v_2) * t$.

Let T be a non empty non void reflexive transitive *TA*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. We say that v is applicable to t if and only if the condition (Def. 21) is satisfied.

(Def. 21) Let i be a natural number, a be an adjective of T, and s be a type of T. If $i \in \operatorname{dom} v$ and $a = v(i)$ and $s = (\operatorname{apply}(v, t))(i)$, then a is applicable to s.

One can prove the following propositions:

(39) Let T be a non empty non void reflexive transitive *TA*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is applicable to t.

(40) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and a be an adjective of T. Then a is applicable to t if and only if $\langle a \rangle$ is applicable to t.

(41) Let T be a non empty non void reflexive transitive *TA*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose $v_1 \frown v_2$ is applicable to t. Then v_1 is applicable to t and v_2 is applicable to $v_1 * t$.

(42) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let i_1, i_2 be natural numbers. Suppose $1 \leq i_1$ and $i_1 \leq i_2$ and $i_2 \leq \operatorname{len} v + 1$. Let t_1, t_2 be types of T. If $t_1 = (\operatorname{apply}(v, t))(i_1)$ and $t_2 = (\operatorname{apply}(v, t))(i_2)$, then $t_2 \leq t_1$.

(43) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let s be a type of T. If $s \in \operatorname{rng} \operatorname{apply}(v, t)$, then $v * t \leq s$ and $s \leq t$.

(44) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is applicable to t, then $v * t \leq t$.

(45) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is applicable to t, then $\operatorname{rng} v \subseteq \operatorname{adjs}(v * t)$.

(46) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let A be a subset of the set of adjectives of T. If $A = \operatorname{rng} v$, then A is applicable to t.

(47) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose v_1 is applicable to t and $\operatorname{rng} v_2 \subseteq \operatorname{rng} v_1$. Let s be a type of T. If $s \in \operatorname{rng} \operatorname{apply}(v_2, t)$, then $v_1 * t \leq s$.

(48) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. If $v_1 \frown v_2$ is applicable to t, then $v_2 \frown v_1$ is applicable to t.

(49) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. If $v_1 \frown v_2$ is applicable to t, then $(v_1 \frown v_2) * t = (v_2 \frown v_1) * t$.

(50) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A * t \leq t$.

(51) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A \subseteq \operatorname{adjs}(A * t)$.

(52) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If A is applicable to t, then $A * t \in \operatorname{types} A$.

(53) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, A be a subset of the set of adjectives of T, and t' be a type of T. If $t' \leq t$ and $A \subseteq \operatorname{adjs} t'$, then A is applicable to t and $t' \leq A * t$.

(54) Let T be a Noetherian reflexive transitive antisymmetric *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. If $A \subseteq \operatorname{adjs} t$, then A is applicable to t and $A * t = t$.

(55) Let T be a *TA*-structure, t be a type of T, and A, B be subsets of the set of adjectives of T. If A is applicable to t and $B \subseteq A$, then B is applicable to t.

(56) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, a be an adjective of T, and A, B be subsets of the set of adjectives of T. If $B = A \cup \{a\}$ and B is applicable to t, then $a * (A * t) = B * t$.

(57) Let T be a Noetherian reflexive transitive antisymmetric non void *TA*-structure with l.u.b.'s with structured adjectives, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. Suppose v is applicable to t. Let A be a subset of the set of adjectives of T. If $A = \operatorname{rng} v$, then $v * t = A * t$.

4. Subject Function

Let T be a non empty non void *TA*-structure. The functor $\operatorname{sub} T$ yielding a function from the set of adjectives of T into the carrier of T is defined by:

(Def. 22) For every adjective a of T holds $(\operatorname{sub} T)(a) = \sup(\operatorname{types} a \cup \operatorname{types} \operatorname{non} a)$.

We consider *TAS*-structures as extensions of *TA*-structure as systems
⟨ a carrier, a set of adjectives, an internal relation, an operation non, an adjective map, a subject map ⟩,

where the carrier and the set of adjectives are sets, the internal relation is a binary relation on the carrier, the operation non is a unary operation on the set of adjectives, the adjective map is a function from the carrier into Fin the set of adjectives, and the subject map is a function from the set of adjectives into the carrier.

One can verify that there exists a *TAS*-structure which is non void, reflexive, trivial, non empty, and strict.

Let T be a non empty non void *TAS*-structure and let a be an adjective of T. The functor $\mathrm{sub}\, a$ yielding a type of T is defined as follows:

(Def. 23) $\mathrm{sub}\, a = (\text{the subject map of } T)(a)$.

Let T be a non empty non void *TAS*-structure. We say that T is absorbing non if and only if:

(Def. 24) $(\text{The subject map of } T) \cdot (\text{the operation non of } T) = \text{the subject map of } T$.

We say that T is subjected if and only if:

(Def. 25) For every adjective a of T holds $\mathrm{types}\, a \cup \mathrm{types}\, \mathrm{non}\, a \leq \mathrm{sub}\, a$ and if $\mathrm{types}\, a \neq \emptyset$ and $\mathrm{types}\, \mathrm{non}\, a \neq \emptyset$, then $\mathrm{sub}\, a = \sup(\mathrm{types}\, a \cup \mathrm{types}\, \mathrm{non}\, a)$.

Let T be a non empty non void *TAS*-structure. Let us observe that T is absorbing non if and only if:

(Def. 26) For every adjective a of T holds $\mathrm{sub}\, \mathrm{non}\, a = \mathrm{sub}\, a$.

Let T be a non empty non void *TAS*-structure, let t be an element of T, and let a be an adjective of T. We say that a is properly applicable to t if and only if:

(Def. 27) $t \leq \mathrm{sub}\, a$ and a is applicable to t.

Let T be a non empty non void reflexive transitive *TAS*-structure, let t be a type of T, and let v be a finite sequence of elements of the set of adjectives of T. We say that v is properly applicable to t if and only if the condition (Def. 28) is satisfied.

(Def. 28) Let i be a natural number, a be an adjective of T, and s be a type of T. If $i \in \mathrm{dom}\, v$ and $a = v(i)$ and $s = (\mathrm{apply}(v,t))(i)$, then a is properly applicable to s.

The following propositions are true:

(58) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v be a finite sequence of elements of the set of adjectives of T. If v is properly applicable to t, then v is applicable to t.

(59) Let T be a non empty non void reflexive transitive *TAS*-structure and t be a type of T. Then $\varepsilon_{(\text{the set of adjectives of } T)}$ is properly applicable to t.

(60) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and a be an adjective of T. Then a is properly applicable to t if and only if $\langle a \rangle$ is properly applicable to t.

(61) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose $v_1 \frown v_2$ is properly applicable to t. Then v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$.

(62) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and v_1, v_2 be finite sequences of elements of the set of adjectives of T. Suppose v_1 is properly applicable to t and v_2 is properly applicable to $v_1 * t$. Then $v_1 \frown v_2$ is properly applicable to t.

Let T be a non empty non void reflexive transitive *TAS*-structure, let t be a type of T, and let A be a subset of the set of adjectives of T. We say that A is properly applicable to t if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exists a finite sequence s of elements of the set of adjectives of T such that $\mathrm{rng}\, s = A$ and s is properly applicable to t.

We now state two propositions:

(63) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and A be a subset of the set of adjectives of T. If A is properly applicable to t, then A is finite.

(64) Let T be a non empty non void reflexive transitive *TAS*-structure and t be a type of T. Then $\emptyset_{\text{the set of adjectives of } T}$ is properly applicable to t.

The scheme *MinimalFiniteSet* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a finite set A such that $\mathcal{P}[A]$ and for every set B such that $B \subseteq A$ and $\mathcal{P}[B]$ holds $B = A$

provided the following condition is met:

- There exists a finite set A such that $\mathcal{P}[A]$.

Next we state the proposition

(65) Let T be a non empty non void reflexive transitive *TAS*-structure, t be a type of T, and A be a subset of the set of adjectives of T. Suppose A is properly applicable to t. Then there exists a subset B of the set of adjectives of T such that

(i) $B \subseteq A$,

(ii) B is properly applicable to t,

(iii) $A * t = B * t$, and

(iv) for every subset C of the set of adjectives of T such that $C \subseteq B$ and C is properly applicable to t and $A * t = C * t$ holds $C = B$.

Let T be a non empty non void reflexive transitive *TAS*-structure. We say that T is commutative if and only if the condition (Def. 30) is satisfied.

(Def. 30) Let t_1, t_2 be types of T and a be an adjective of T. Suppose a is properly applicable to t_1 and $a * t_1 \leq t_2$. Then there exists a finite subset A of the set of adjectives of T such that A is properly applicable to $t_1 \sqcup t_2$ and $A * (t_1 \sqcup t_2) = t_2$.

Let us observe that there exists a complete upper-bounded non empty non void trivial reflexive transitive antisymmetric strict *TAS*-structure which is Mizar-widening-like, involutive, without fixpoints, consistent, absorbing non, subjected, and commutative and has structured adjectives and typed adjectives.

One can prove the following proposition

(66) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a subset of the set of adjectives of T. Suppose A is properly applicable to t. Then there exists an one-to-one finite sequence s of elements of the set of adjectives of T such that $\mathrm{rng}\, s = A$ and s is properly applicable to t.

5. Reduction of Adjectives

Let T be a non empty non void reflexive transitive *TAS*-structure. The functor $\circ\!\!\longrightarrow_T$ yields a binary relation on T and is defined by the condition (Def. 31).

(Def. 31) Let t_1, t_2 be types of T. Then $\langle t_1, t_2 \rangle \in \circ\!\!\longrightarrow_T$ if and only if there exists an adjective a of T such that $a \notin \mathrm{adjs}\, t_2$ and a is properly applicable to t_2 and $a * t_2 = t_1$.

Next we state the proposition

(67) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\longrightarrow_T \subseteq$ the internal relation of T.

The scheme *RedInd* deals with a non empty set $\mathcal{A}$, a binary relation $\mathcal{B}$ on $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

For all elements x, y of $\mathcal{A}$ such that $\mathcal{B}$ reduces x to y holds $\mathcal{P}[x,y]$

provided the parameters have the following properties:

- For all elements x, y of $\mathcal{A}$ such that $\langle x, y\rangle \in \mathcal{B}$ holds $\mathcal{P}[x,y]$,
- For every element x of $\mathcal{A}$ holds $\mathcal{P}[x,x]$, and
- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,z]$ holds $\mathcal{P}[x,z]$.

One can prove the following propositions:

(68) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T. If $\circ\!\!\rightarrow_T$ reduces t_1 to t_2, then $t_1 \leq t_2$.

(69) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T$ is irreflexive.

(70) Let T be an antisymmetric non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T$ is strongly-normalizing.

(71) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T. Suppose $\operatorname{rng} s = A$ and s is properly applicable to t. Let i be a natural number. If $1 \leq i$ and $i \leq \operatorname{len} s$, then $\langle(\operatorname{apply}(s,t))(i+1), (\operatorname{apply}(s,t))(i)\rangle \in \circ\!\!\rightarrow_T$.

(72) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. Suppose that for every subset C of the set of adjectives of T such that $C \subseteq A$ and C is properly applicable to t and $A * t = C * t$ holds $C = A$. Let s be an one-to-one finite sequence of elements of the set of adjectives of T. Suppose $\operatorname{rng} s = A$ and s is properly applicable to t. Then $\operatorname{Rev}(\operatorname{apply}(s,t))$ is a reduction sequence w.r.t. $\circ\!\!\rightarrow_T$.

(73) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives, t be a type of T, and A be a finite subset of the set of adjectives of T. If A is properly applicable to t, then $\circ\!\!\rightarrow_T$ reduces $A * t$ to t.

(74) Let X be a non empty set, R be a binary relation on X, and r be a reduction sequence w.r.t. R. If $r(1) \in X$, then r is a finite sequence of elements of X.

(75) Let X be a non empty set, R be a binary relation on X, x be an element of X, and y be a set. If R reduces x to y, then $y \in X$.

(76) Let X be a non empty set and R be a binary relation on X. Suppose R is weakly-normalizing and has unique normal form property. Let x be an element of X. Then $\operatorname{nf}_R(x) \in X$.

(77) Let T be a Noetherian reflexive transitive antisymmetric non void *TAS*-structure with l.u.b.'s with structured adjectives and t_1, t_2 be types of T. Suppose $\circ\!\!\rightarrow_T$ reduces t_1 to t_2. Then there exists a finite subset A of the set of adjectives of T such that A is properly applicable to t_2 and $t_1 = A * t_2$.

(78) Let T be an antisymmetric commutative non void reflexive transitive Noetherian *TAS*-structure with l.u.b.'s with structured adjectives. Then $\circ\!\!\rightarrow_T$ has Church-Rosser property and unique normal form property.

6. Radix Types

Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and let t be a type of T. The functor $\operatorname{radix} t$ yielding a type of T is defined by:

(Def. 32) $\operatorname{radix} t = \mathrm{nf}_{\circ\!\!\to_T}(t)$.

The following propositions are true:

(79) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t be a type of T. Then $\circ\!\!\to_T$ reduces t to $\operatorname{radix} t$.

(80) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t be a type of T. Then $t \leq \operatorname{radix} t$.

(81) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t be a type of T, and X be a set. Suppose $X = \{t'; t'$ ranges over types of T: $\bigvee_{A:\ \text{finite subset of the set of adjectives of } T}$ (A is properly applicable to $t' \wedge A * t' = t)\}$. Then $\sup X$ exists in T and $\operatorname{radix} t = \bigsqcup_T X$.

(82) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t_1, t_2 be types of T, and a be an adjective of T. If a is properly applicable to t_1 and $a * t_1 \leq \operatorname{radix} t_2$, then $t_1 \leq \operatorname{radix} t_2$.

(83) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s and t_1, t_2 be types of T. If $t_1 \leq t_2$, then $\operatorname{radix} t_1 \leq \operatorname{radix} t_2$.

(84) Let T be an antisymmetric commutative non empty non void reflexive transitive Noetherian *TAS*-structure with structured adjectives and l.u.b.'s, t be a type of T, and a be an adjective of T. If a is properly applicable to t, then $\operatorname{radix}(a * t) = \operatorname{radix} t$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[5] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[6] Grzegorz Bancerek. Reduction relations. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/rewrite1.html`.

[7] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_0.html`.

[8] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_0.html`.

[9] Grzegorz Bancerek. On the structure of Mizar types. In Herman Geuvers and Fairouz Kamareddine, editors, *Electronic Notes in Theoretical Computer Science*, volume 85. Elsevier, 2003.

[10] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[11] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[12] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[13] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[14] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_4.html`.

[15] Czesław Byliński. Some properties of restrictions of finite sequences. *Journal of Formalized Mathematics*, 7, 1995. `http://mizar.org/JFM/Vol7/finseq_5.html`.

[16] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[17] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_1.html`.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[19] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finsub_1.html`.

[20] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_1.html`.

[21] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[22] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[23] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

[24] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_2.html`.

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