JOURNAL OF FORMALIZED MATHEMATICS
Volume 3, Released 1991, Published 2003
Inst. of Computer Science, Univ. of Białystok

Mostowski's Fundamental Operations — Part II

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Summary. The article consists of two parts. The first part is translation of chapter II.3 of [13]. A section of $D_H(a)$ determined by f (symbolically $S_H(a,f)$) and a notion of predicative closure of a class are defined. It is proved that if following assumptions are satisfied: (o) $A = \bigcup_\xi A_\xi$, (i) $A_\xi \subset A_\eta$ for $\xi < \eta$, (ii) $A_\lambda = \bigcup_{\xi<\lambda} A_\lambda$ (λ is a limit number), (iii) $A_\xi \in A$, (iv) A_ξ is transitive, (v) $(x,y \in A) \to (x \cap y \in A)$, (vi) A is predicatively closed, then the axiom of power sets and the axiom of substitution are valid in A. The second part is continuation of [12]. It is proved that if a non-void, transitive class is closed with respect to the operations $A_1 - A_7$ then it is predicatively closed. At last sufficient criteria for a class to be a model of ZF-theory are formulated: if A_ξ satisfies o – iv and A is closed under the operations $A_1 - A_7$ then A is a model of ZF.

MML Identifier: ZF_FUND2.

WWW: http://mizar.org/JFM/Vol3/zf_fund2.html

The articles [17], [16], [11], [20], [18], [21], [9], [10], [4], [2], [3], [5], [1], [14], [8], [15], [6], [7], [12], and [19] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: H is a ZF-formula, M, E are non empty sets, e is an element of E, m is an element of M, v is a function from VAR into M, and f is a function from VAR into E.

Let us consider H, M, v. The functor $\mathrm{S}_v(H)$ yields a subset of M and is defined by:

(Def. 1) $$\mathrm{S}_v(H) = \begin{cases} \{m : M, v(\tfrac{\mathrm{x}_0}{m}) \models H\}, & \text{if } \mathrm{x}_0 \in \mathrm{Free}\, H, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let us consider M. We say that M is predicatively closed if and only if:

(Def. 2) For all H, E, f such that $E \in M$ holds $\mathrm{S}_f(H) \in M$.

We now state the proposition

(1) If E is transitive, then $\mathrm{S}_{f(\frac{\mathrm{x}_1}{e})}(\forall_{\mathrm{x}_2}(\mathrm{x}_2 \varepsilon(\mathrm{x}_0) \Rightarrow \mathrm{x}_2 \varepsilon(\mathrm{x}_1))) = E \cap 2^e$.

For simplicity, we use the following convention: W denotes a universal class, Y denotes a subclass of W, a, b denote ordinals of W, and L denotes a transfinite sequence of non empty sets from W.

One can prove the following propositions:

(2) Suppose for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive and $\bigcup L$ is predicatively closed. Then $\bigcup L \models$ the axiom of power sets.

(3) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$,

(iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$,

(iv) for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive, and

(v) $\bigcup L$ is predicatively closed.

Let given H. If $\{\mathrm{x}_0, \mathrm{x}_1, \mathrm{x}_2\}$ misses $\mathrm{Free}\, H$, then $\bigcup L \models$ the axiom of substitution for H.

(4) $\mathrm{S}_v(H) = \{m : \{\langle \emptyset, m \rangle\} \cup (v \cdot \mathrm{decode}) \restriction (\mathrm{code}(\mathrm{Free}\, H) \setminus \{\emptyset\}) \in \mathrm{D}_M(H)\}$.

(5) If Y is closed w.r.t. A1-A7 and transitive, then Y is predicatively closed.

(6) Suppose that

(i) $\omega \in W$,

(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$,

(iii) for every a such that $a \neq \emptyset$ and a is a limit ordinal number holds $L(a) = \bigcup(L \restriction a)$,

(iv) for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive, and

(v) $\bigcup L$ is closed w.r.t. A1-A7.

Then $\bigcup L$ is a model of ZF.

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Received February 15, 1991

Published January 2, 2004