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On Same Equivalents of Well-foundedness[1]

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Summary. Four statements equivalent to well-foundedness (well-founded induction, existence of recursively defined functions, uniqueness of recursively defined functions, and absence of descending ω-chains) have been proved in Mizar and the proofs were mechanically checked for correctness. It seems not to be widely known that the existence (without the uniqueness assumption) of recursively defined functions implies well-foundedness. In the proof we used regular cardinals, a fairly advanced notion of set theory. This work was inspired by T. Franzen's paper [14]. Franzen's proofs were written by a mathematician having an argument with a computer scientist. We were curious about the effort needed to formalize Franzen's proofs given the state of the Mizar Mathematical Library at that time (July 1996). The formalization went quite smoothly once the mathematics was sorted out.

MML Identifier: `WELLFND1`.

WWW: `http://mizar.org/JFM/Vol9/wellfnd1.html`

The articles [19], [12], [23], [21], [2], [24], [9], [16], [25], [11], [10], [18], [3], [5], [4], [13], [1], [22], [20], [8], [17], [6], [15], and [7] provide the notation and terminology for this paper.

1. Preliminaries

Let A, B be sets. One can verify that $A \dot{\to} B$ is functional.

Let R be a 1-sorted structure, let X be a set, and let p be a partial function from the carrier of R to X. Then $\operatorname{dom} p$ is a subset of R.

We now state two propositions:

(1) For every set X and for all functions f, g such that $f \subseteq g$ and $X \subseteq \operatorname{dom} f$ holds $f \restriction X = g \restriction X$.

(2) Let X be a functional set. Suppose that for all functions f, g such that $f \in X$ and $g \in X$ holds $f \approx g$. Then $\bigcup X$ is a function.

The scheme *PFSeparation* deals with sets $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a subset P_1 of $\mathcal{A} \dot{\to} \mathcal{B}$ such that for every partial function p_1 from $\mathcal{A}$ to $\mathcal{B}$ holds $p_1 \in P_1$ iff $\mathcal{P}[p_1]$

for all values of the parameters.

Let X be a set. Note that X^+ is non empty.

One can check that there exists an aleph which is regular.

The following two propositions are true:

(3) For every regular aleph M and for every set X such that $X \subseteq M$ and $\overline{\overline{X}} \in M$ holds $\sup X \in M$.

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(4) For every relational structure R and for every set x holds (the internal relation of R)-Seg(x) $\subseteq$ the carrier of R.

Let R be a relational structure and let X be a subset of R. Let us observe that X is lower if and only if:

(Def. 1) For all sets x, y such that $x \in X$ and $\langle y, x \rangle \in$ the internal relation of R holds $y \in X$.

The following propositions are true:

(5) Let R be a relational structure, X be a subset of R, and x be a set. If X is lower and $x \in X$, then (the internal relation of R)-Seg(x) $\subseteq X$.

(6) Let R be a relational structure, X be a lower subset of R, Y be a subset of R, and x be a set. If $Y = X \cup \{x\}$ and (the internal relation of R)-Seg(x) $\subseteq X$, then Y is lower.

2. Well Founded Relational Structures

Let R be a relational structure. We say that R is well founded if and only if:

(Def. 2) The internal relation of R is well founded in the carrier of R.

One can check that there exists a relational structure which is non empty and well founded.

Let R be a relational structure and let X be a subset of R. We say that X is well founded if and only if:

(Def. 3) The internal relation of R is well founded in X.

Let R be a relational structure. Note that there exists a subset of R which is well founded.

Let R be a relational structure. The functor WF-Part(R) yielding a subset of R is defined as follows:

(Def. 4) WF-Part(R) $= \bigcup \{S; S$ ranges over subsets of R: S is well founded and lower$\}$.

Let R be a relational structure. Observe that WF-Part(R) is lower and well founded.

One can prove the following propositions:

(7) For every non empty relational structure R and for every element x of R holds $\{x\}$ is a well founded subset of R.

(8) Let R be a relational structure and X, Y be well founded subsets of R. If X is lower, then $X \cup Y$ is a well founded subset of R.

(9) For every relational structure R holds R is well founded iff WF-Part(R) = the carrier of R.

(10) Let R be a non empty relational structure and x be an element of R. If (the internal relation of R)-Seg(x) $\subseteq$ WF-Part(R), then $x \in$ WF-Part(R).

The scheme *WFMin* deals with a non empty relational structure $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists an element x of $\mathcal{A}$ such that $\mathcal{P}[x]$ and it is not true that there exists an element y of $\mathcal{A}$ such that $x \neq y$ and $\mathcal{P}[y]$ and $\langle y, x \rangle \in$ the internal relation of $\mathcal{A}$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\mathcal{B}]$, and
- $\mathcal{A}$ is well founded.

Next we state the proposition

(11) Let R be a non empty relational structure. Then R is well founded if and only if for every set S such that for every element x of R such that (the internal relation of R)-Seg(x) $\subseteq S$ holds $x \in S$ holds the carrier of $R \subseteq S$.

The scheme *WFInduction* deals with a non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element x of $\mathcal{A}$ holds $\mathcal{P}[x]$

provided the parameters meet the following conditions:

- Let x be an element of $\mathcal{A}$. Suppose that for every element y of $\mathcal{A}$ such that $y \neq x$ and $\langle y, x \rangle \in$ the internal relation of $\mathcal{A}$ holds $\mathcal{P}[y]$. Then $\mathcal{P}[x]$, and
- $\mathcal{A}$ is well founded.

Let R be a non empty relational structure, let V be a non empty set, let H be a function from $[\!:$the carrier of R, (the carrier of R)$\dot{\rightarrow}V:\!]$ into V, and let F be a function. We say that F is recursively expressed by H if and only if:

(Def. 5) For every element x of R holds $F(x) = H(\langle x, F{\restriction}$(the internal relation of R)-Seg$(x)\rangle)$.

One can prove the following three propositions:

(12) Let R be a non empty relational structure. Then R is well founded if and only if for every non empty set V and for every function H from $[\!:$the carrier of R, (the carrier of R)$\dot{\rightarrow}V:\!]$ into V holds there exists a function from the carrier of R into V which is recursively expressed by H.

(13) Let R be a non empty relational structure and V be a non trivial set. Suppose that for every function H from $[\!:$the carrier of R, (the carrier of R)$\dot{\rightarrow}V:\!]$ into V and for all functions F_1, F_2 from the carrier of R into V such that F_1 is recursively expressed by H and F_2 is recursively expressed by H holds $F_1 = F_2$. Then R is well founded.

(14) Let R be a non empty well founded relational structure, V be a non empty set, H be a function from $[\!:$the carrier of R, (the carrier of R)$\dot{\rightarrow}V:\!]$ into V, and F_1, F_2 be functions from the carrier of R into V. Suppose F_1 is recursively expressed by H and F_2 is recursively expressed by H. Then $F_1 = F_2$.

Let R be a relational structure and let f be a sequence of R. We say that f is descending if and only if:

(Def. 7)[1] For every natural number n holds $f(n+1) \neq f(n)$ and $\langle f(n+1), f(n) \rangle \in$ the internal relation of R.

One can prove the following proposition

(15) For every non empty relational structure R holds R is well founded iff there exists no sequence of R which is descending.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek. Sequences of ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal2.html`.

[5] Grzegorz Bancerek. The well ordering relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/wellord1.html`.

[6] Grzegorz Bancerek. On powers of cardinals. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/card_5.html`.

[7] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_0.html`.

[8] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/realset1.html`.

[1] The definition (Def. 6) has been removed.

[9] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[10] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[11] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[12] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[13] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[14] T. Franzen. Teaching mathematics through formalism: a few caveats. In D. Gries, editor, *Proceedings of the DIMACS Symposium on Teaching Logic*. DIMACS, 1996.
On WWW: http://dimacs.rutgers.edu/Workshops/Logic/program.html.

[15] Jarosław Kotowicz and Yuji Sakai. Properties of partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/rfunct_3.html.

[16] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[17] Jan Popiołek. Real normed space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/normsp_1.html.

[18] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funcop_1.html.

[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[20] Andrzej Trybulec. Function domains and Frænkel operator. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/fraenkel.html.

[21] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[22] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/orders_1.html.

[23] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[24] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[25] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

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