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Inst. of Computer Science, Univ. of Białystok

Algebraic Lattices

Robert Milewski
Warsaw University
Białystok

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The articles [16], [8], [18], [14], [10], [19], [17], [7], [1], [15], [2], [3], [12], [20], [4], [9], [6], [11], [13], and [5] provide the notation and terminology for this paper.

1. The Subset of All Compact Elements

Let L be a non empty reflexive relational structure. The functor CompactSublatt(L) yielding a strict full relational substructure of L is defined as follows:

(Def. 1) For every element x of L holds $x \in$ the carrier of CompactSublatt(L) iff x is compact.

Let L be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that CompactSublatt(L) is non empty.

We now state three propositions:

(1) Let L be a complete lattice and x, y, k be elements of L. If $x \leq k$ and $k \leq y$ and $k \in$ the carrier of CompactSublatt(L), then $x \ll y$.

(2) Let L be a complete lattice and x be an element of L. Then $\uparrow x$ is an open filter of L if and only if x is compact.

(3) For every lower-bounded non empty poset L with l.u.b.'s holds CompactSublatt(L) is join-inheriting and $\perp_L \in$ the carrier of CompactSublatt(L).

Let L be a non empty reflexive relational structure and let x be an element of L. The functor compactbelow(x) yielding a subset of L is defined by:

(Def. 2) compactbelow(x) $= \{y; y$ ranges over elements of L: $x \geq y \wedge y$is compact$\}$.

The following propositions are true:

(4) Let L be a non empty reflexive relational structure and x, y be elements of L. Then $y \in$ compactbelow(x) if and only if the following conditions are satisfied:

(i) $x \geq y$, and

(ii) y is compact.

(5) For every non empty reflexive relational structure L and for every element x of L holds compactbelow(x) $= \downarrow x \cap$ the carrier of CompactSublatt(L).

(6) For every non empty reflexive transitive relational structure L and for every element x of L holds compactbelow$(x) \subseteq {\downarrow}x$.

Let L be a non empty lower-bounded reflexive antisymmetric relational structure and let x be an element of L. Observe that compactbelow(x) is non empty.

2. Algebraic Lattices

Let L be a non empty reflexive relational structure. We say that L satisfies axiom K if and only if:

(Def. 3) For every element x of L holds $x = \sup$ compactbelow(x).

Let L be a non empty reflexive relational structure. We say that L is algebraic if and only if:

(Def. 4) For every element x of L holds compactbelow(x) is non empty and directed and L is up-complete and satisfies axiom K.

Next we state the proposition

(7) Let L be a lattice. Then L is algebraic if and only if the following conditions are satisfied:

(i) L is continuous, and

(ii) for all elements x, y of L such that $x \ll y$ there exists an element k of L such that $k \in$ the carrier of CompactSublatt(L) and $x \leq k$ and $k \leq y$.

Let us observe that every lattice which is algebraic is also continuous.

Let us mention that every non empty reflexive relational structure which is algebraic is also up-complete and satisfies axiom K.

Let L be a non empty poset with l.u.b.'s. Note that CompactSublatt(L) is join-inheriting.

Let L be a non empty reflexive relational structure. We say that L is arithmetic if and only if:

(Def. 5) L is algebraic and CompactSublatt(L) is meet-inheriting.

3. Arithmetic Lattices

Let us note that every lattice which is arithmetic is also algebraic.

Let us note that every lattice which is trivial is also arithmetic.

One can verify that there exists a lattice which is trivial and strict.

Next we state a number of propositions:

(8) Let L_1, L_2 be non empty reflexive antisymmetric relational structures. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is up-complete. Let x_1, y_1 be elements of L_1 and x_2, y_2 be elements of L_2. If $x_1 = x_2$ and $y_1 = y_2$ and $x_1 \ll y_1$, then $x_2 \ll y_2$.

(9) Let L_1, L_2 be non empty reflexive antisymmetric relational structures. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is up-complete. Let x be an element of L_1 and y be an element of L_2. If $x = y$ and x is compact, then y is compact.

(10) Let L_1, L_2 be up-complete non empty reflexive antisymmetric relational structures. Suppose the relational structure of L_1 = the relational structure of L_2. Let x be an element of L_1 and y be an element of L_2. If $x = y$, then compactbelow(x) = compactbelow(y).

(11) Let L_1, L_2 be relational structures. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is non empty. Then L_2 is non empty.

(12) Let L_1, L_2 be non empty relational structures. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is reflexive. Then L_2 is reflexive.

(13) Let L_1, L_2 be relational structures. Suppose the relational structure of L_1 = the relational structure of L_2 and L_1 is transitive. Then L_2 is transitive.

(14) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is antisymmetric. Then L_2 is antisymmetric.

(15) Let L_1, L_2 be non empty reflexive relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is up-complete. Then L_2 is up-complete.

(16) Let L_1, L_2 be up-complete non empty reflexive antisymmetric relational structures such that the relational structure of $L_1 =$ the relational structure of L_2 and L_1 satisfies axiom K and for every element x of L_1 holds compactbelow(x) is non empty and directed. Then L_2 satisfies axiom K.

(17) Let L_1, L_2 be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is algebraic. Then L_2 is algebraic.

(18) Let L_1, L_2 be lattices. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is arithmetic. Then L_2 is arithmetic.

Let L be a non empty relational structure. Note that the relational structure of L is non empty.

Let L be a non empty reflexive relational structure. Observe that the relational structure of L is reflexive.

Let L be a transitive relational structure. One can verify that the relational structure of L is transitive.

Let L be an antisymmetric relational structure. Observe that the relational structure of L is antisymmetric.

Let L be a relational structure with g.l.b.'s. Observe that the relational structure of L has g.l.b.'s.

Let L be a relational structure with l.u.b.'s. One can check that the relational structure of L has l.u.b.'s.

Let L be an up-complete non empty reflexive relational structure. Note that the relational structure of L is up-complete.

Let L be an algebraic non empty reflexive antisymmetric relational structure. Note that the relational structure of L is algebraic.

Let L be an arithmetic lattice. One can check that the relational structure of L is arithmetic.

We now state several propositions:

(19) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and X be a subset of S. Suppose sup X exists in L and $\bigsqcup_L X$ is an element of S. Then sup X exists in S and $\sup X = \bigsqcup_L X$.

(20) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and X be a subset of S. Suppose inf X exists in L and $\bigsqcap_L X$ is an element of S. Then inf X exists in S and $\inf X = \bigsqcap_L X$.

(21) For every algebraic lattice L holds L is arithmetic iff CompactSublatt(L) is a lattice.

(22) For every algebraic lower-bounded lattice L holds L is arithmetic iff $\ll_L$ is multiplicative.

(23) Let L be an arithmetic lower-bounded lattice and p be an element of L. If p is pseudoprime, then p is prime.

(24) Let L be an algebraic distributive lower-bounded lattice. Suppose that for every element p of L such that p is pseudoprime holds p is prime. Then L is arithmetic.

Let L be an algebraic lattice and let c be a closure map from L into L. Note that there exists a subset of $\mathrm{Im}\, c$ which is non empty and directed.

The following propositions are true:

(25) Let L be an algebraic lattice and c be a closure map from L into L. If c is directed-sups-preserving, then $c^{\circ}(\Omega_{\mathrm{CompactSublatt}(L)}) \subseteq \Omega_{\mathrm{CompactSublatt}(\mathrm{Im}\, c)}$.

(26) Let L be an algebraic lower-bounded lattice and c be a closure map from L into L. If c is directed-sups-preserving, then $\mathrm{Im}\, c$ is an algebraic lattice.

(27) Let L be an algebraic lower-bounded lattice and c be a closure map from L into L. If c is directed-sups-preserving, then $c^{\circ}(\Omega_{\mathrm{CompactSublatt}(L)}) = \Omega_{\mathrm{CompactSublatt}(\mathrm{Im}\, c)}$.

4. Boolean Posets as Algebraic Lattices

One can prove the following propositions:

(28) For all sets X, x holds x is an element of $2^X_\subseteq$ iff $x \subseteq X$.

(29) Let X be a set and x, y be elements of $2^X_\subseteq$. Then $x \ll y$ if and only if for every family Y of subsets of X such that $y \subseteq \bigcup Y$ there exists a finite subset Z of Y such that $x \subseteq \bigcup Z$.

(30) For every set X and for every element x of $2^X_\subseteq$ holds x is finite iff x is compact.

(31) For every set X and for every element x of $2^X_\subseteq$ holds compactbelow$(x) = \{y : y$ ranges over finite subsets of $x\}$.

(32) For every set X and for every subset F of X holds $F \in$ the carrier of CompactSublatt$(2^X_\subseteq)$ iff F is finite.

Let X be a set and let x be an element of $2^X_\subseteq$. One can verify that compactbelow(x) is lower and directed.

We now state the proposition

(33) For every set X holds $2^X_\subseteq$ is algebraic.

Let X be a set. Note that $2^X_\subseteq$ is algebraic.

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