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# Meet Continuous Lattices Revisited[1]

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**Summary.** This work is a continuation of formalization of [13]. Theorems from Chapter III, Section 2, pp. 153–156 are proved.

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The articles [23], [9], [29], [30], [31], [7], [8], [12], [28], [22], [32], [11], [21], [1], [24], [2], [3], [14], [10], [15], [16], [17], [4], [25], [26], [20], [18], [27], [5], [6], and [19] provide the notation and terminology for this paper.

The following propositions are true:

(1) For every set $x$ and for every non empty set $D$ holds $x \cap \bigcup D = \bigcup \{x \cap d : d$ ranges over elements of $D\}$.

(2) Let $R$ be a non empty reflexive transitive relational structure and $D$ be a non empty directed subset of $\langle \mathrm{Ids}(R), \subseteq \rangle$. Then $\bigcup D$ is an ideal of $R$.

Let $R$ be a non empty reflexive transitive relational structure. Observe that $\langle \mathrm{Ids}(R), \subseteq \rangle$ is up-complete.

Next we state two propositions:

(3) Let $R$ be a non empty reflexive transitive relational structure and $D$ be a non empty directed subset of $\langle \mathrm{Ids}(R), \subseteq \rangle$. Then $\sup D = \bigcup D$.

(4) Let $R$ be a semilattice, $D$ be a non empty directed subset of $\langle \mathrm{Ids}(R), \subseteq \rangle$, and $x$ be an element of $\langle \mathrm{Ids}(R), \subseteq \rangle$. Then $\sup(\{x\} \sqcap D) = \bigcup \{x \cap d : d$ ranges over elements of $D\}$.

Let $R$ be a semilattice. Observe that $\langle \mathrm{Ids}(R), \subseteq \rangle$ satisfies MC.

Let $R$ be a non empty trivial relational structure. Observe that every topological augmentation of $R$ is trivial.

We now state three propositions:

(5) Let $S$ be a Scott complete top-lattice, $T$ be a complete lattice, and $A$ be a Scott topological augmentation of $T$. Suppose the relational structure of $S =$ the relational structure of $T$. Then the FR-structure of $A =$ the FR-structure of $S$.

(6) Let $N$ be a Lawson complete top-lattice, $T$ be a complete lattice, and $A$ be a Lawson correct topological augmentation of $T$. Suppose the relational structure of $N =$ the relational structure of $T$. Then the FR-structure of $A =$ the FR-structure of $N$.

(7) Let $N$ be a Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, $A$ be a subset of $N$, and $J$ be a subset of $S$. If $A = J$ and $J$ is closed, then $A$ is closed.

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Let $A$ be a complete lattice. Note that $\lambda(A)$ is non empty.

Let $S$ be a Scott complete top-lattice. Note that $\langle \sigma(S), \subseteq \rangle$ is complete and non trivial.

Let $N$ be a Lawson complete top-lattice. Observe that $\langle \sigma(N), \subseteq \rangle$ is complete and non trivial and $\langle \lambda(N), \subseteq \rangle$ is complete and non trivial.

Next we state several propositions:

(8) Let $T$ be a non empty reflexive relational structure. Then $\sigma(T) \subseteq \{W \setminus \uparrow F; W$ ranges over subsets of $T$, $F$ ranges over subsets of $T$: $W \in \sigma(T) \wedge F$ is finite$\}$.

(9) For every Lawson complete top-lattice $N$ holds $\lambda(N) =$ the topology of $N$.

(10) For every Lawson complete top-lattice $N$ holds $\sigma(N) \subseteq \lambda(N)$.

(11) Let $M$, $N$ be complete lattices. Suppose the relational structure of $M =$ the relational structure of $N$. Then $\lambda(M) = \lambda(N)$.

(12) For every Lawson complete top-lattice $N$ and for every subset $X$ of $N$ holds $X \in \lambda(N)$ iff $X$ is open.

Let us note that every reflexive non empty FR-structure which is trivial and topological space-like is also Scott.

Let us note that every complete top-lattice which is trivial is also Lawson.

One can verify that there exists a complete top-lattice which is strict, continuous, lower-bounded, meet-continuous, and Scott.

Let us observe that there exists a complete top-lattice which is strict, continuous, compact, Hausdorff, and Lawson.

We now state the proposition

(13) Let $N$ be a meet-continuous lattice and $A$ be a subset of $N$. If $A$ has the property (S), then $\uparrow A$ has the property (S).

Let $N$ be a meet-continuous lattice and let $A$ be a property(S) subset of $N$. Observe that $\uparrow A$ is property(S).

One can prove the following propositions:

(14) Let $N$ be a meet-continuous Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, and $A$ be a subset of $N$. If $A \in \lambda(N)$, then $\uparrow A \in \sigma(S)$.

(15) Let $N$ be a meet-continuous Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, $A$ be a subset of $N$, and $J$ be a subset of $S$. If $A = J$, then if $A$ is open, then $\uparrow J$ is open.

(16) Let $N$ be a meet-continuous Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, $x$ be a point of $S$, $y$ be a point of $N$, and $J$ be a basis of $y$. If $x = y$, then $\{\uparrow A; A$ ranges over subsets of $N$: $A \in J\}$ is a basis of $x$.

(17) Let $N$ be a meet-continuous Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, $X$ be an upper subset of $N$, and $Y$ be a subset of $S$. If $X = Y$, then $\mathrm{Int}\, X = \mathrm{Int}\, Y$.

(18) Let $N$ be a meet-continuous Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, $X$ be a lower subset of $N$, and $Y$ be a subset of $S$. If $X = Y$, then $\overline{X} = \overline{Y}$.

(19) Let $M$, $N$ be complete lattices, $L_1$ be a Lawson correct topological augmentation of $M$, and $L_2$ be a Lawson correct topological augmentation of $N$. Suppose $\langle \sigma(N), \subseteq \rangle$ is continuous. Then the topology of $[: L_1, (L_2$ **qua** topological space$) :] = \lambda([: M, N :])$.

(20) Let $M$, $N$ be complete lattices, $P$ be a Lawson correct topological augmentation of $[: M, N :]$, $Q$ be a Lawson correct topological augmentation of $M$, and $R$ be a Lawson correct topological augmentation of $N$. Suppose $\langle \sigma(N), \subseteq \rangle$ is continuous. Then the topological structure of $P = [: Q, (R$ **qua** topological space$) :]$.

(21) For every meet-continuous Lawson complete top-lattice $N$ and for every element $x$ of $N$ holds $x \sqcap \square$ is continuous.

Let $N$ be a meet-continuous Lawson complete top-lattice and let $x$ be an element of $N$. Observe that $x \sqcap \square$ is continuous.

One can prove the following two propositions:

(22) Let $N$ be a meet-continuous Lawson complete top-lattice such that $\langle \sigma(N), \subseteq \rangle$ is continuous. Then $N$ satisfies conditions of topological semilattice.

(23) Let $N$ be a meet-continuous Lawson complete top-lattice. Suppose $\langle \sigma(N), \subseteq \rangle$ is continuous. Then $N$ is Hausdorff if and only if for every subset $X$ of $[:N, (N$ **qua** topological space$):]$ such that $X =$ the internal relation of $N$ holds $X$ is closed.

Let $N$ be a non empty reflexive relational structure and let $X$ be a subset of $N$. The functor $X^0$ yielding a subset of $N$ is defined as follows:

(Def. 1) $X^0 = \{u; u$ ranges over elements of $N$: $\bigwedge_{D:\text{ non empty directed subset of } N} (u \leq \sup D \Rightarrow X$ meets $D)\}$.

Let $N$ be a non empty reflexive antisymmetric relational structure and let $X$ be an empty subset of $N$. Note that $X^0$ is empty.

The following propositions are true:

(24) For every non empty reflexive relational structure $N$ and for all subsets $A$, $J$ of $N$ such that $A \subseteq J$ holds $A^0 \subseteq J^0$.

(25) For every non empty reflexive relational structure $N$ and for every element $x$ of $N$ holds $\uparrow x^0 = \Uparrow x$.

(26) For every Scott top-lattice $N$ and for every upper subset $X$ of $N$ holds $\mathrm{Int} X \subseteq X^0$.

(27) For every non empty reflexive relational structure $N$ and for all subsets $X$, $Y$ of $N$ holds $X^0 \cup Y^0 \subseteq X \cup Y^0$.

(28) For every meet-continuous lattice $N$ and for all upper subsets $X$, $Y$ of $N$ holds $X^0 \cup Y^0 = X \cup Y^0$.

(29) Let $S$ be a meet-continuous Scott top-lattice and $F$ be a finite subset of $S$. Then $\mathrm{Int} \uparrow F \subseteq \bigcup \{\Uparrow x; x$ ranges over elements of $S$: $x \in F\}$.

(30) Let $N$ be a Lawson complete top-lattice. Then $N$ is continuous if and only if $N$ is meet-continuous and Hausdorff.

Let us mention that every complete top-lattice which is continuous and Lawson is also Hausdorff and every complete top-lattice which is meet-continuous, Lawson, and Hausdorff is also continuous.

Let $N$ be a non empty FR-structure. We say that $N$ has small semilattices if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let $x$ be a point of $N$. Then there exists a generalized basis $J$ of $x$ such that for every subset $A$ of $N$ if $A \in J$, then $\mathrm{sub}(A)$ is meet-inheriting.

We say that $N$ has compact semilattices if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let $x$ be a point of $N$. Then there exists a generalized basis $J$ of $x$ such that for every subset $A$ of $N$ if $A \in J$, then $\mathrm{sub}(A)$ is meet-inheriting and $A$ is compact.

We say that $N$ has open semilattices if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let $x$ be a point of $N$. Then there exists a basis $J$ of $x$ such that for every subset $A$ of $N$ if $A \in J$, then $\mathrm{sub}(A)$ is meet-inheriting.

One can check the following observations:

* $*$ every non empty topological space-like FR-structure which has open semilattices has also small semilattices,
* $*$ every non empty topological space-like FR-structure which has compact semilattices has also small semilattices,
* $*$ every non empty FR-structure which is anti-discrete has also small semilattices and open semilattices, and
* $*$ every non empty FR-structure which is reflexive, trivial, and topological space-like has also compact semilattices.

Let us note that there exists a top-lattice which is strict, trivial, and lower.
Next we state several propositions:

(31) Let $N$ be top-poset with g.l.b.'s satisfying conditions of topological semilattice and $C$ be a subset of $N$. If $\mathrm{sub}(C)$ is meet-inheriting, then $\mathrm{sub}(\overline{C})$ is meet-inheriting.

(32) Let $N$ be a meet-continuous Lawson complete top-lattice and $S$ be a Scott topological augmentation of $N$. Then for every point $x$ of $S$ there exists a basis $J$ of $x$ such that for every subset $W$ of $S$ such that $W \in J$ holds $W$ is a filter of $S$ if and only if $N$ has open semilattices.

(33) Let $N$ be a Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, and $x$ be an element of $N$. Then $\{\inf A; A$ ranges over subsets of $S$: $x \in A \wedge A \in \sigma(S)\} \subseteq \{\inf J; J$ ranges over subsets of $N$: $x \in J \wedge J \in \lambda(N)\}$.

(34) Let $N$ be a meet-continuous Lawson complete top-lattice, $S$ be a Scott topological augmentation of $N$, and $x$ be an element of $N$. Then $\{\inf A; A$ ranges over subsets of $S$: $x \in A \wedge A \in \sigma(S)\} = \{\inf J; J$ ranges over subsets of $N$: $x \in J \wedge J \in \lambda(N)\}$.

(35) Let $N$ be a meet-continuous Lawson complete top-lattice. Then $N$ is continuous if and only if $N$ has open semilattices and $\langle \sigma(N), \subseteq \rangle$ is continuous.

Let us observe that every Lawson complete top-lattice which is continuous has also open semilattices.

Let $N$ be a continuous Lawson complete top-lattice. One can verify that $\langle \sigma(N), \subseteq \rangle$ is continuous.

One can prove the following propositions:

(36) Every continuous Lawson complete top-lattice is compact and Hausdorff, has open semilattices, and satisfies conditions of topological semilattice.

(37) Every Hausdorff Lawson complete top-lattice with open semilattices and satisfying conditions of topological semilattice has compact semilattices.

(38) Let $N$ be a meet-continuous Hausdorff Lawson complete top-lattice and $x$ be an element of $N$. Then $x = \bigsqcup_N \{\inf V; V$ ranges over subsets of $N$: $x \in V \wedge V \in \lambda(N)\}$.

(39) Let $N$ be a meet-continuous Lawson complete top-lattice. Then $N$ is continuous if and only if for every element $x$ of $N$ holds $x = \bigsqcup_N \{\inf V; V$ ranges over subsets of $N$: $x \in V \wedge V \in \lambda(N)\}$.

(40) Let $N$ be a meet-continuous Lawson complete top-lattice. Then $N$ is algebraic if and only if $N$ has open semilattices and $\langle \sigma(N), \subseteq \rangle$ is algebraic.

Let $N$ be a meet-continuous algebraic Lawson complete top-lattice. Observe that $\langle \sigma(N), \subseteq \rangle$ is algebraic.

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