JOURNAL OF FORMALIZED MATHEMATICS
Volume 10, Released 1998, Published 2003
Inst. of Computer Science, Univ. of Białystok

Kernel Projections and Quotient Lattices

Piotr Rudnicki
University of Alberta
Edmonton

Summary. This article completes the Mizar formalization of Chapter I, Section 2 from [13]. After presenting some preliminary material (not all of which is later used in this article) we give the proof of theorem 2.7 (i), p.60. We do not follow the hint from [13] suggesting using the equations 2.3, p. 58. The proof is taken directly from the definition of continuous lattice. The goal of the last section is to prove the correspondence between the set of all congruences of a continuous lattice and the set of all kernel operators of the lattice which preserve directed sups (Corollary 2.13).

MML Identifier: WAYBEL20.

WWW: http://mizar.org/JFM/Vol10/waybel20.html

The articles [21], [10], [24], [25], [26], [17], [27], [7], [9], [8], [12], [20], [19], [22], [6], [1], [23], [2], [18], [3], [14], [28], [15], [4], [11], [5], and [16] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) For every set X and for every subset S of id_X holds $\pi_1(S) = \pi_2(S)$.

(2) For all non empty sets X, Y and for every function f from X into Y holds $[: f, f :]^{-1}(\mathrm{id}_Y)$ is an equivalence relation of X.

Let L_1, L_2, T_1, T_2 be relational structures, let f be a map from L_1 into T_1, and let g be a map from L_2 into T_2. Then $[: f, g :]$ is a map from $[: L_1, L_2 :]$ into $[: T_1, T_2 :]$.

The following propositions are true:

(3) For all functions f, g and for every set X holds $\pi_1([: f, g :]^\circ X) \subseteq f^\circ \pi_1(X)$ and $\pi_2([: f, g :]^\circ X) \subseteq g^\circ \pi_2(X)$.

(4) For all functions f, g and for every set X such that $X \subseteq [: \mathrm{dom} f, \mathrm{dom} g :]$ holds $\pi_1([: f, g :]^\circ X) = f^\circ \pi_1(X)$ and $\pi_2([: f, g :]^\circ X) = g^\circ \pi_2(X)$.

(5) For every non empty antisymmetric relational structure S such that $\inf \emptyset$ exists in S holds S is upper-bounded.

(6) For every non empty antisymmetric relational structure S such that $\sup \emptyset$ exists in S holds S is lower-bounded.

(7) Let L_1, L_2 be antisymmetric non empty relational structures and D be a subset of $[: L_1, L_2 :]$. If $\inf D$ exists in $[: L_1, L_2 :]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D) \rangle$.

(8) Let L_1, L_2 be antisymmetric non empty relational structures and D be a subset of $[:L_1, L_2:]$. If sup D exists in $[:L_1, L_2:]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.

(9) Let L_1, L_2, T_1, T_2 be antisymmetric non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is infs-preserving and g is infs-preserving. Then $[:f, g:]$ is infs-preserving.

(10) Let L_1, L_2, T_1, T_2 be antisymmetric reflexive non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is filtered-infs-preserving and g is filtered-infs-preserving. Then $[:f, g:]$ is filtered-infs-preserving.

(11) Let L_1, L_2, T_1, T_2 be antisymmetric non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is sups-preserving and g is sups-preserving. Then $[:f, g:]$ is sups-preserving.

(12) Let L_1, L_2, T_1, T_2 be antisymmetric reflexive non empty relational structures, f be a map from L_1 into T_1, and g be a map from L_2 into T_2. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $[:f, g:]$ is directed-sups-preserving.

(13) Let L be an antisymmetric non empty relational structure and X be a subset of $[:L, L:]$. Suppose $X \subseteq \mathrm{id}_{\text{the carrier of } L}$ and inf X exists in $[:L, L:]$. Then $\inf X \in \mathrm{id}_{\text{the carrier of } L}$.

(14) Let L be an antisymmetric non empty relational structure and X be a subset of $[:L, L:]$. Suppose $X \subseteq \mathrm{id}_{\text{the carrier of } L}$ and sup X exists in $[:L, L:]$. Then $\sup X \in \mathrm{id}_{\text{the carrier of } L}$.

(15) Let L, M be non empty relational structures. If L and M are isomorphic and L is reflexive, then M is reflexive.

(16) Let L, M be non empty relational structures. If L and M are isomorphic and L is transitive, then M is transitive.

(17) Let L, M be non empty relational structures. Suppose L and M are isomorphic and L is antisymmetric. Then M is antisymmetric.

(18) Let L, M be non empty relational structures. If L and M are isomorphic and L is complete, then M is complete.

(19) Let L be a non empty transitive relational structure and k be a map from L into L. If k is infs-preserving, then k° is infs-preserving.

(20) Let L be a non empty transitive relational structure and k be a map from L into L. If k is filtered-infs-preserving, then k° is filtered-infs-preserving.

(21) Let L be a non empty transitive relational structure and k be a map from L into L. If k is sups-preserving, then k° is sups-preserving.

(22) Let L be a non empty transitive relational structure and k be a map from L into L. If k is directed-sups-preserving, then k° is directed-sups-preserving.

(24)[1] Let S, T be reflexive antisymmetric non empty relational structures and f be a map from S into T. If f is filtered-infs-preserving, then f is monotone.

(25) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is monotone. Let X be a subset of S. If X is filtered, then $f^\circ X$ is filtered.

(26) Let L_1, L_2, L_3 be non empty relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is infs-preserving and g is infs-preserving. Then $g \cdot f$ is infs-preserving.

[1] The proposition (23) has been removed.

(27) Let L_1, L_2, L_3 be non empty reflexive antisymmetric relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is filtered-infs-preserving and g is filtered-infs-preserving. Then $g \cdot f$ is filtered-infs-preserving.

(28) Let L_1, L_2, L_3 be non empty relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is sups-preserving and g is sups-preserving. Then $g \cdot f$ is sups-preserving.

(29) Let L_1, L_2, L_3 be non empty reflexive antisymmetric relational structures, f be a map from L_1 into L_2, and g be a map from L_2 into L_3. Suppose f is directed-sups-preserving and g is directed-sups-preserving. Then $g \cdot f$ is directed-sups-preserving.

2. Some Remarks on Lattice Product

Next we state several propositions:

(30) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a lower-bounded antisymmetric relational structure. Then $\prod J$ is lower-bounded.

(31) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is an upper-bounded antisymmetric relational structure. Then $\prod J$ is upper-bounded.

(32) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a lower-bounded antisymmetric relational structure. Let i be an element of I. Then $\bot_{\prod J}(i) = \bot_{J(i)}$.

(33) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is an upper-bounded antisymmetric relational structure. Let i be an element of I. Then $\top_{\prod J}(i) = \top_{J(i)}$.

(34) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a continuous complete lattice. Then $\prod J$ is continuous.

3. Kernel Projections and Quotient Lattices

Next we state the proposition

(35) Let L, T be continuous complete lattices, g be a CLHomomorphism of L, T, and S be a subset of $[: L, L :]$. Suppose $S = [: g, g :]^{-1}(\mathrm{id}_{\text{the carrier of } T})$. Then $\mathrm{sub}(S)$ is a continuous subframe of $[: L, L :]$.

Let L be a relational structure and let R be a subset of $[: L, L :]$. Let us assume that R is an equivalence relation of the carrier of L. The functor $\mathrm{EqRel}(R)$ yielding an equivalence relation of the carrier of L is defined by:

(Def. 1) $\mathrm{EqRel}(R) = R$.

Let L be a non empty relational structure and let R be a subset of $[: L, L :]$. We say that R is continuous lattice congruence if and only if:

(Def. 2) R is an equivalence relation of the carrier of L and $\mathrm{sub}(R)$ is a continuous subframe of $[: L, L :]$.

Next we state the proposition

(36) Let L be a complete lattice and R be a non empty subset of $[: L, L :]$. Suppose R is continuous lattice congruence. Let x be an element of L. Then $\langle \inf([x]_{\mathrm{EqRel}(R)}), x \rangle \in R$.

Let L be a complete lattice and let R be a non empty subset of $[:L, L:]$. Let us assume that R is continuous lattice congruence. The kernel operation of R yielding a kernel map from L into L is defined by:

(Def. 3) For every element x of L holds (the kernel operation of $R)(x) = \inf([x]_{\mathrm{EqRel}(R)})$.

One can prove the following three propositions:

(37) Let L be a complete lattice and R be a non empty subset of $[:L, L:]$. Suppose R is continuous lattice congruence. Then

(i) the kernel operation of R is directed-sups-preserving, and

(ii) $R = [:$the kernel operation of R, the kernel operation of $R:]^{-1}(\mathrm{id}_{\text{the carrier of } L})$.

(38) Let L be a continuous complete lattice, R be a subset of $[:L, L:]$, and k be a kernel map from L into L. Suppose k is directed-sups-preserving and $R = [:k, k:]^{-1}(\mathrm{id}_{\text{the carrier of } L})$. Then there exists a continuous complete strict lattice L_4 such that

(i) the carrier of $L_4 = \mathrm{Classes}\,\mathrm{EqRel}(R)$,

(ii) the internal relation of $L_4 = \{\langle [x]_{\mathrm{EqRel}(R)}, [y]_{\mathrm{EqRel}(R)}\rangle; x$ ranges over elements of L, y ranges over elements of L: $k(x) \leq k(y)\}$, and

(iii) for every map g from L into L_4 such that for every element x of L holds $g(x) = [x]_{\mathrm{EqRel}(R)}$ holds g is a CLHomomorphism of L, L_4.

(39) Let L be a continuous complete lattice and R be a subset of $[:L, L:]$. Suppose that

(i) R is an equivalence relation of the carrier of L, and

(ii) there exists a continuous complete lattice L_4 such that the carrier of $L_4 = \mathrm{Classes}\,\mathrm{EqRel}(R)$ and for every map g from L into L_4 such that for every element x of L holds $g(x) = [x]_{\mathrm{EqRel}(R)}$ holds g is a CLHomomorphism of L, L_4.

Then $\mathrm{sub}(R)$ is a continuous subframe of $[:L, L:]$.

Let L be a non empty reflexive relational structure. Note that there exists a map from L into L which is directed-sups-preserving and kernel.

Let L be a non empty reflexive relational structure and let k be a kernel map from L into L. The kernel congruence of k yielding a non empty subset of $[:L, L:]$ is defined as follows:

(Def. 4) The kernel congruence of $k = [:k, k:]^{-1}(\mathrm{id}_{\text{the carrier of } L})$.

We now state two propositions:

(40) Let L be a non empty reflexive relational structure and k be a kernel map from L into L. Then the kernel congruence of k is an equivalence relation of the carrier of L.

(41) Let L be a continuous complete lattice and k be a directed-sups-preserving kernel map from L into L. Then the kernel congruence of k is continuous lattice congruence.

Let L be a continuous complete lattice and let R be a non empty subset of $[:L, L:]$. Let us assume that R is continuous lattice congruence. The functor ${}^{L}/_{R}$ yields a continuous complete strict lattice and is defined as follows:

(Def. 5) The carrier of ${}^{L}/_{R} = \mathrm{Classes}\,\mathrm{EqRel}(R)$ and for all elements x, y of ${}^{L}/_{R}$ holds $x \leq y$ iff $\bigsqcap_L x \leq \bigsqcap_L y$.

We now state four propositions:

(42) Let L be a continuous complete lattice and R be a non empty subset of $[:L, L:]$. Suppose R is continuous lattice congruence. Let x be a set. Then x is an element of ${}^{L}/_{R}$ if and only if there exists an element y of L such that $x = [y]_{\mathrm{EqRel}(R)}$.

(43) Let L be a continuous complete lattice and R be a non empty subset of $[:L, L:]$. Suppose R is continuous lattice congruence. Then $R =$ the kernel congruence of the kernel operation of R.

(44) Let L be a continuous complete lattice and k be a directed-sups-preserving kernel map from L into L. Then $k =$ the kernel operation of the kernel congruence of k.

(45) Let L be a continuous complete lattice and p be a projection map from L into L. Suppose p is infs-preserving. Then $\operatorname{Im} p$ is a continuous lattice and $\operatorname{Im} p$ is infs-inheriting.

REFERENCES

[1] Grzegorz Bancerek. Curried and uncurried functions. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/funct_5.html`.

[2] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[3] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_0.html`.

[4] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_0.html`.

[5] Grzegorz Bancerek. The "way-below" relation. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_3.html`.

[6] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_3.html`.

[7] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[8] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[9] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[10] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[11] Czesław Byliński. Galois connections. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/waybel_1.html`.

[12] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[13] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[14] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_1.html`.

[15] Artur Korniłowicz. Cartesian products of relations and relational structures. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_3.html`.

[16] Robert Milewski. Completely-irreducible elements. *Journal of Formalized Mathematics*, 10, 1998. `http://mizar.org/JFM/Vol10/waybel16.html`.

[17] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[19] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/eqrel_1.html`.

[20] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[22] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/pboole.html`.

[23] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/orders_1.html`.

[24] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[25] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[26] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

[27] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_2.html`.

[28] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Journal of Formalized Mathematics*, 8, 1996. `http://mizar.org/JFM/Vol8/yellow_2.html`.

Received July 6, 1998

Published January 2, 2004