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König's Lemma

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Summary. A continuation of [3]. The notion of finite–order trees, succesors of an element of a tree, and chains, levels and branches of a tree are introduced. That notion has been used to formalize König's Lemma which claims that there is a infinite branch of a finite-order tree if the tree has arbitrary long finite chains. Besides, the concept of decorated trees is introduced and some concepts dealing with trees are applied to decorated trees.

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The articles [11], [8], [13], [4], [14], [6], [2], [12], [5], [9], [1], [7], [10], and [3] provide the notation and terminology for this paper.

For simplicity, we follow the rules: x, y, X are sets, W, W_1, W_2 are trees, w is an element of W, f is a function, D, D' are non empty sets, k, k_1, k_2, m, n are natural numbers, v, v_1, v_2 are finite sequences, and p, q, r are finite sequences of elements of $\mathbb{N}$.

Next we state four propositions:

(1) For all v_1, v_2, v such that $v_1 \preceq v$ and $v_2 \preceq v$ holds v_1 and v_2 are $\subseteq$-comparable.

(2) For all v_1, v_2, v such that $v_1 \prec v$ and $v_2 \preceq v$ holds v_1 and v_2 are $\subseteq$-comparable.

(4)[1] If $\operatorname{len} v_1 = k+1$, then there exist v_2, x such that $v_1 = v_2 \frown \langle x \rangle$ and $\operatorname{len} v_2 = k$.

(6)[2] $\operatorname{Seg}_{\preceq}(v \frown \langle x \rangle) = \operatorname{Seg}_{\preceq}(v) \cup \{v\}$.

The scheme *TreeStruct Ind* deals with a tree $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every element t of $\mathcal{A}$ holds $\mathcal{P}[t]$

provided the following conditions are satisfied:

- $\mathcal{P}[\emptyset]$, and
- For every element t of $\mathcal{A}$ and for every n such that $\mathcal{P}[t]$ and $t \frown \langle n \rangle \in \mathcal{A}$ holds $\mathcal{P}[t \frown \langle n \rangle]$.

We now state the proposition

(7) If for every p holds $p \in W_1$ iff $p \in W_2$, then $W_1 = W_2$.

Let us consider W_1, W_2. Let us observe that $W_1 = W_2$ if and only if:

(Def. 1) For every p holds $p \in W_1$ iff $p \in W_2$.

The following propositions are true:

[1] The proposition (3) has been removed.
[2] The proposition (5) has been removed.

(8) If $p \in W$, then $W = W$ with-replacement$(p, W \restriction p)$.

(9) If $p \in W$ and $q \in W$ and $p \not\preceq q$, then $q \in W$ with-replacement(p, W_1).

(10) If $p \in W$ and $q \in W$ and p and q are not $\subseteq$-comparable, then W with-replacement(p, W_1) with-replacement$(q, W_2) = W$ with-replacement(q, W_2) with-replacement(p, W_1).

Let I_1 be a tree. We say that I_1 is finite-order if and only if:

(Def. 2) There exists n such that for every element t of I_1 holds $t \frown \langle n \rangle \notin I_1$.

One can check that there exists a tree which is finite-order.

Let us consider W. A subset of W is called a chain of W if:

(Def. 3) For all p, q such that $p \in$ it and $q \in$ it holds p and q are $\subseteq$-comparable.

A subset of W is called a level of W if:

(Def. 4) There exists n such that it $= \{w : \operatorname{len} w = n\}$.

Let us consider w. The functor $\operatorname{succ} w$ yields a subset of W and is defined as follows:

(Def. 5) $\operatorname{succ} w = \{w \frown \langle n \rangle : w \frown \langle n \rangle \in W\}$.

Next we state three propositions:

(11) Every level of W is an antichain of prefixes of W.

(12) $\operatorname{succ} w$ is an antichain of prefixes of W.

(13) For every antichain A of prefixes of W and for every chain C of W there exists w such that $A \cap C \subseteq \{w\}$.

Let us consider W, n. The functor W-level(n) yielding a level of W is defined as follows:

(Def. 6) W-level$(n) = \{w : \operatorname{len} w = n\}$.

We now state several propositions:

(14) $w \frown \langle n \rangle \in \operatorname{succ} w$ iff $w \frown \langle n \rangle \in W$.

(15) If $w = \emptyset$, then W-level$(1) = \operatorname{succ} w$.

(16) $W = \bigcup\{W\text{-level}(n)\}$.

(17) For every finite tree W holds $W = \bigcup\{W\text{-level}(n) : n \leq \operatorname{height} W\}$.

(18) For every level L of W there exists n such that $L = W$-level(n).

Now we present two schemes. The scheme *FraenkelCard* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\overline{\overline{\{\mathcal{F}(w); w \text{ ranges over elements of } \mathcal{A} : w \in \mathcal{B}\}}} \leq \overline{\overline{\mathcal{B}}}$

for all values of the parameters.

The scheme *FraenkelFinCard* deals with a non empty set $\mathcal{A}$, finite sets $\mathcal{B}$, $\mathcal{C}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

$\operatorname{card} \mathcal{C} \leq \operatorname{card} \mathcal{B}$

provided the parameters satisfy the following condition:

- $\mathcal{C} = \{\mathcal{F}(w); w \text{ ranges over elements of } \mathcal{A} : w \in \mathcal{B}\}$.

We now state two propositions:

(19) If W is finite-order, then there exists n such that for every w there exists a finite set B such that $B = \operatorname{succ} w$ and $\operatorname{card} B \leq n$.

(20) If W is finite-order, then $\operatorname{succ} w$ is finite.

Let W be a finite-order tree and let w be an element of W. One can check that $\operatorname{succ} w$ is finite.
We now state two propositions:

(21) $\emptyset$ is a chain of W.

(22) $\{\emptyset\}$ is a chain of W.

Let us consider W. Observe that there exists a chain of W which is non empty.
Let us consider W and let I_1 be a chain of W. We say that I_1 is branch-like if and only if:

(Def. 7) For every p such that $p \in I_1$ holds $\mathrm{Seg}_{\preceq}(p) \subseteq I_1$ and it is not true that there exists p such that $p \in W$ and for every q such that $q \in I_1$ holds $q \prec p$.

Let us consider W. Note that there exists a chain of W which is branch-like.
Let us consider W. A branch of W is a branch-like chain of W.
Let us consider W. One can verify that every chain of W which is branch-like is also non empty.
In the sequel C denotes a chain of W and B denotes a branch of W.
The following two propositions are true:

(23) If $v_1 \in C$ and $v_2 \in C$, then $v_1 \in \mathrm{Seg}_{\preceq}(v_2)$ or $v_2 \preceq v_1$.

(24) If $v_1 \in C$ and $v_2 \in C$ and $v \preceq v_2$, then $v_1 \in \mathrm{Seg}_{\preceq}(v)$ or $v \preceq v_1$.

Let us consider W. One can check that there exists a chain of W which is finite.
Next we state several propositions:

(25) For every finite chain C of W such that $\operatorname{card} C > n$ there exists p such that $p \in C$ and $\operatorname{len} p \geq n$.

(26) For every C holds $\{w : \bigvee_p \ (p \in C \ \wedge \ w \preceq p)\}$ is a chain of W.

(27) If $p \preceq q$ and $q \in B$, then $p \in B$.

(28) $\emptyset \in B$.

(29) If $p \in C$ and $q \in C$ and $\operatorname{len} p \leq \operatorname{len} q$, then $p \preceq q$.

(30) There exists B such that $C \subseteq B$.

Now we present two schemes. The scheme *FuncExOfMinNat* deals with a set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ there exists n such that $f(x) = n$ and $\mathcal{P}[x,n]$ and for every m such that $\mathcal{P}[x,m]$ holds $n \leq m$

provided the parameters satisfy the following condition:

- For every x such that $x \in \mathcal{A}$ there exists n such that $\mathcal{P}[x,n]$.

The scheme *InfiniteChain* deals with a set $\mathcal{A}$, a set $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

There exists f such that $\operatorname{dom} f = \mathbb{N}$ and $\operatorname{rng} f \subseteq \mathcal{A}$ and $f(0) = \mathcal{B}$ and for every k holds $\mathcal{Q}[f(k), f(k+1)]$ and $\mathcal{P}[f(k)]$

provided the following requirements are met:

- $\mathcal{B} \in \mathcal{A}$ and $\mathcal{P}[\mathcal{B}]$, and
- For every x such that $x \in \mathcal{A}$ and $\mathcal{P}[x]$ there exists y such that $y \in \mathcal{A}$ and $\mathcal{Q}[x,y]$ and $\mathcal{P}[y]$.

We now state two propositions:

(31) Let T be a tree. Suppose for every n there exists a finite chain C of T such that $\operatorname{card} C = n$ and for every element t of T holds $\operatorname{succ} t$ is finite. Then there exists a chain B of T such that B is not finite.

(32) Let T be a finite-order tree. Suppose that for every n there exists a finite chain C of T such that $\operatorname{card} C = n$. Then there exists a chain B of T such that B is not finite.

Let I_1 be a binary relation. We say that I_1 is decorated tree-like if and only if:

(Def. 8) $\operatorname{dom} I_1$ is a tree.

Let us mention that there exists a function which is decorated tree-like.

A decorated tree is a decorated tree-like function.

In the sequel T, T_1, T_2 denote decorated trees.

Let us consider T. One can verify that $\operatorname{dom} T$ is non empty and tree-like.

Let X be a set. A binary relation is called a ParametrizedSubset of X if:

(Def. 9) $\operatorname{rng} \mathrm{it} \subseteq X$.

Let us consider D. One can verify that there exists a ParametrizedSubset of D which is decorated tree-like and function-like.

Let us consider D. A tree decorated with elements of D is a decorated tree-like function-like ParametrizedSubset of D.

Let D be a non empty set, let T be a tree decorated with elements of D, and let t be an element of $\operatorname{dom} T$. Then $T(t)$ is an element of D.

The following proposition is true

(33) If $\operatorname{dom} T_1 = \operatorname{dom} T_2$ and for every p such that $p \in \operatorname{dom} T_1$ holds $T_1(p) = T_2(p)$, then $T_1 = T_2$.

Now we present two schemes. The scheme *DTreeEx* deals with a tree $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists T such that $\operatorname{dom} T = \mathcal{A}$ and for every p such that $p \in \mathcal{A}$ holds $\mathcal{P}[p, T(p)]$

provided the following condition is met:

- For every p such that $p \in \mathcal{A}$ there exists x such that $\mathcal{P}[p, x]$.

The scheme *DTreeLambda* deals with a tree $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists T such that $\operatorname{dom} T = \mathcal{A}$ and for every p such that $p \in \mathcal{A}$ holds $T(p) = \mathcal{F}(p)$

for all values of the parameters.

Let us consider T. The functor $\operatorname{Leaves}(T)$ yielding a set is defined by:

(Def. 10) $\operatorname{Leaves}(T) = T^{\circ} \operatorname{Leaves}(\operatorname{dom} T)$.

Let us consider p. The functor $T \restriction p$ yielding a decorated tree is defined by:

(Def. 11) $\operatorname{dom}(T \restriction p) = \operatorname{dom} T \restriction p$ and for every q such that $q \in \operatorname{dom} T \restriction p$ holds $(T \restriction p)(q) = T(p \frown q)$.

Next we state the proposition

(34) If $p \in \operatorname{dom} T$, then $\operatorname{rng}(T \restriction p) \subseteq \operatorname{rng} T$.

Let us consider D and let T be a tree decorated with elements of D. Then $\operatorname{Leaves}(T)$ is a subset of D. Let p be an element of $\operatorname{dom} T$. Then $T \restriction p$ is a tree decorated with elements of D.

Let us consider T, p, T_1. Let us assume that $p \in \operatorname{dom} T$. The functor T with-replacement(p, T_1) yields a decorated tree and is defined by the conditions (Def. 12).

(Def. 12)(i) $\operatorname{dom}(T \text{ with-replacement}(p, T_1)) = \operatorname{dom} T \text{ with-replacement}(p, \operatorname{dom} T_1)$, and

(ii) for every q such that $q \in \operatorname{dom} T \text{ with-replacement}(p, \operatorname{dom} T_1)$ holds $p \not\preceq q$ and $(T \text{ with-replacement}(p, T_1))(q) = T(q)$ or there exists r such that $r \in \operatorname{dom} T_1$ and $q = p \frown r$ and $(T \text{ with-replacement}(p, T_1))(q) = T_1(r)$.

Let us consider W, x. One can verify that $W \longmapsto x$ is decorated tree-like.

Let D be a non empty set, let us consider W, and let d be an element of D. Then $W \longmapsto d$ is a tree decorated with elements of D.

We now state four propositions:

(35) If for every x such that $x \in D$ holds x is a tree, then $\bigcup D$ is a tree.

(36) Suppose for every x such that $x \in X$ holds x is a function and X is $\subseteq$-linear. Then $\bigcup X$ is relation-like and function-like.

(37) Suppose for every x such that $x \in D$ holds x is a decorated tree and D is $\subseteq$-linear. Then $\bigcup D$ is a decorated tree.

(38) Suppose for every x such that $x \in D'$ holds x is a tree decorated with elements of D and D' is $\subseteq$-linear. Then $\bigcup D'$ is a tree decorated with elements of D.

Now we present two schemes. The scheme *DTreeStructEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a function $\mathcal{C}$ from $[:\mathcal{A}, \mathbb{N}:]$ into $\mathcal{A}$, and states that:

There exists a tree T decorated with elements of $\mathcal{A}$ such that
(i) $T(\emptyset) = \mathcal{B}$, and
(ii) for every element t of $\operatorname{dom} T$ holds $\operatorname{succ} t = \{t \frown \langle k \rangle : k \in \mathcal{F}(T(t))\}$ and for all n, x such that $x = T(t)$ and $n \in \mathcal{F}(x)$ holds $T(t \frown \langle n \rangle) = \mathcal{C}(\langle x, n \rangle)$

provided the following requirement is met:

- For every element d of $\mathcal{A}$ and for all k_1, k_2 such that $k_1 \leq k_2$ and $k_2 \in \mathcal{F}(d)$ holds $k_1 \in \mathcal{F}(d)$.

The scheme *DTreeStructFinEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a natural number, and a function $\mathcal{C}$ from $[:\mathcal{A}, \mathbb{N}:]$ into $\mathcal{A}$, and states that:

There exists a tree T decorated with elements of $\mathcal{A}$ such that
(i) $T(\emptyset) = \mathcal{B}$, and
(ii) for every element t of $\operatorname{dom} T$ holds $\operatorname{succ} t = \{t \frown \langle k \rangle : k < \mathcal{F}(T(t))\}$ and for all n, x such that $x = T(t)$ and $n < \mathcal{F}(x)$ holds $T(t \frown \langle n \rangle) = \mathcal{C}(\langle x, n \rangle)$

for all values of the parameters.

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