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The Brouwer Fixed Point Theorem for Intervals[1]

Toshihiko Watanabe
Shinshu University
Nagano

Summary. The aim is to prove, using Mizar System, the following simplest version of the Brouwer Fixed Point Theorem [3]. *For every continuous mapping* $f : \mathbb{I} \to \mathbb{I}$ *of the topological unit interval* $\mathbb{I}$ *there exists a point x such that* $f(x) = x$ (see e.g. [9], [4]).

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The articles [17], [20], [1], [19], [21], [5], [6], [10], [16], [15], [7], [14], [11], [13], [2], [8], [12], and [18] provide the notation and terminology for this paper.

1. Properties of Topological Intervals

In this paper a, b, c, d are real numbers.

One can prove the following propositions:

(1) If $a \leq c$ and $d \leq b$, then $[c,d] \subseteq [a,b]$.

(2) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a,b] \cup [c,d] = [a,d]$.

(3) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a,b] \cap [c,d] = [c,b]$.

(4) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = [a,b]$ holds A is closed.

(5) If $a \leq b$, then $[a,\, b]_{\rm T}$ is a closed subspace of $\mathbb{R}^{\mathbf{1}}$.

(6) If $a \leq c$ and $d \leq b$ and $c \leq d$, then $[c,\, d]_{\rm T}$ is a closed subspace of $[a,\, b]_{\rm T}$.

(7) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a,\, d]_{\rm T} = [a,\, b]_{\rm T} \cup [c,\, d]_{\rm T}$ and $[c,\, b]_{\rm T} = [a,\, b]_{\rm T} \cap [c,\, d]_{\rm T}$.

Let a, b be real numbers. Let us assume that $a \leq b$. The functor $a_{[a,b]_{\rm T}}$ yields a point of $[a,\, b]_{\rm T}$ and is defined as follows:

(Def. 1) $a_{[a,b]_{\rm T}} = a$.

The functor $b_{[a,b]_{\rm T}}$ yields a point of $[a,\, b]_{\rm T}$ and is defined by:

(Def. 2) $b_{[a,b]_{\rm T}} = b$.

One can prove the following two propositions:

(8) $0_{\mathbb{I}} = 0_{[0,1]_{\rm T}}$ and $1_{\mathbb{I}} = 1_{[0,1]_{\rm T}}$.

(9) If $a \leq b$ and $b \leq c$, then $a_{[a,b]_{\rm T}} = a_{[a,c]_{\rm T}}$ and $c_{[b,c]_{\rm T}} = c_{[a,c]_{\rm T}}$.

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2. Continuous Mappings Between Topological Intervals

Let a, b be real numbers. Let us assume that $a \leq b$. Let t_1, t_2 be points of $[a, b]_{\rm T}$. The functor $\mathrm{L}_{01}(t_1,t_2)$ yielding a map from $[0, 1]_{\rm T}$ into $[a, b]_{\rm T}$ is defined as follows:

(Def. 3) For every point s of $[0, 1]_{\rm T}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $(\mathrm{L}_{01}(t_1,t_2))(s) = (1-r)\cdot r_1 + r\cdot r_2$.

Next we state four propositions:

(10) Suppose $a \leq b$. Let t_1, t_2 be points of $[a, b]_{\rm T}$, s be a point of $[0, 1]_{\rm T}$, and r, r_1, r_2 be real numbers. If $s = r$ and $r_1 = t_1$ and $r_2 = t_2$, then $(\mathrm{L}_{01}(t_1,t_2))(s) = (r_2 - r_1)\cdot r + r_1$.

(11) If $a \leq b$, then for all points t_1, t_2 of $[a, b]_{\rm T}$ holds $\mathrm{L}_{01}(t_1,t_2)$ is a continuous map from $[0, 1]_{\rm T}$ into $[a, b]_{\rm T}$.

(12) If $a \leq b$, then for all points t_1, t_2 of $[a, b]_{\rm T}$ holds $(\mathrm{L}_{01}(t_1,t_2))(0_{[0,1]_{\rm T}}) = t_1$ and $(\mathrm{L}_{01}(t_1,t_2))(1_{[0,1]_{\rm T}}) = t_2$.

(13) $\mathrm{L}_{01}(0_{[0,1]_{\rm T}}, 1_{[0,1]_{\rm T}}) = \mathrm{id}_{[0,1]_{\rm T}}$.

Let a, b be real numbers. Let us assume that $a < b$. Let t_1, t_2 be points of $[0, 1]_{\rm T}$. The functor $\mathrm{P}_{01}(a,b,t_1,t_2)$ yielding a map from $[a, b]_{\rm T}$ into $[0, 1]_{\rm T}$ is defined by the condition (Def. 4).

(Def. 4) Let s be a point of $[a, b]_{\rm T}$ and r, r_1, r_2 be real numbers. If $s = r$ and $r_1 = t_1$ and $r_2 = t_2$, then $(\mathrm{P}_{01}(a,b,t_1,t_2))(s) = \frac{(b-r)\cdot r_1 + (r-a)\cdot r_2}{b-a}$.

Next we state several propositions:

(14) Suppose $a < b$. Let t_1, t_2 be points of $[0, 1]_{\rm T}$, s be a point of $[a, b]_{\rm T}$, and r, r_1, r_2 be real numbers. If $s = r$ and $r_1 = t_1$ and $r_2 = t_2$, then $(\mathrm{P}_{01}(a,b,t_1,t_2))(s) = \frac{r_2 - r_1}{b-a}\cdot r + \frac{b\cdot r_1 - a\cdot r_2}{b-a}$.

(15) If $a < b$, then for all points t_1, t_2 of $[0, 1]_{\rm T}$ holds $\mathrm{P}_{01}(a,b,t_1,t_2)$ is a continuous map from $[a, b]_{\rm T}$ into $[0, 1]_{\rm T}$.

(16) If $a < b$, then for all points t_1, t_2 of $[0, 1]_{\rm T}$ holds $(\mathrm{P}_{01}(a,b,t_1,t_2))(a_{[a,b]_{\rm T}}) = t_1$ and $(\mathrm{P}_{01}(a,b,t_1,t_2))(b_{[a,b]_{\rm T}}) = t_2$.

(17) $\mathrm{P}_{01}(0,1,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}}) = \mathrm{id}_{[0,1]_{\rm T}}$.

(18) If $a < b$, then $\mathrm{id}_{[a,b]_{\rm T}} = \mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}}) \cdot \mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}})$ and $\mathrm{id}_{[0,1]_{\rm T}} = \mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}}) \cdot \mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}})$.

(19) If $a < b$, then $\mathrm{id}_{[a,b]_{\rm T}} = \mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}}) \cdot \mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}})$ and $\mathrm{id}_{[0,1]_{\rm T}} = \mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}}) \cdot \mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}})$.

(20) Suppose $a < b$. Then $\mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}})$ is a homeomorphism and $(\mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}}))^{-1} = \mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}})$ and $\mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}})$ is a homeomorphism and $(\mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}}))^{-1} = \mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}})$.

(21) Suppose $a < b$. Then $\mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}})$ is a homeomorphism and $(\mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}}))^{-1} = \mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}})$ and $\mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}})$ is a homeomorphism and $(\mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}}))^{-1} = \mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}})$.

3. CONNECTEDNESS OF INTERVALS AND BROUWER FIXED POINT THEOREM FOR INTERVALS

We now state several propositions:

(22) $\mathbb{I}$ is connected.

(23) If $a \leq b$, then $[a, b]_{\mathrm{T}}$ is connected.

(24) For every continuous map f from $\mathbb{I}$ into $\mathbb{I}$ there exists a point x of $\mathbb{I}$ such that $f(x) = x$.

(25) If $a \leq b$, then for every continuous map f from $[a, b]_{\mathrm{T}}$ into $[a, b]_{\mathrm{T}}$ there exists a point x of $[a, b]_{\mathrm{T}}$ such that $f(x) = x$.

(26) Let X, Y be non empty subspaces of $\mathbb{R}^{\mathbf{1}}$ and f be a continuous map from X into Y. Given real numbers a, b such that $a \leq b$ and $[a,b] \subseteq$ the carrier of X and $[a,b] \subseteq$ the carrier of Y and $f^{\circ}[a,b] \subseteq [a,b]$. Then there exists a point x of X such that $f(x) = x$.

(27) Let X, Y be non empty subspaces of $\mathbb{R}^{\mathbf{1}}$ and f be a continuous map from X into Y. Given real numbers a, b such that $a \leq b$ and $[a,b] \subseteq$ the carrier of X and $f^{\circ}[a,b] \subseteq [a,b]$. Then there exists a point x of X such that $f(x) = x$.

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