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Intermediate Value Theorem and Thickness of Simple Closed Curves

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Summary. Various types of the intermediate value theorem ([14]) are proved. For their special cases, the Bolzano theorem is also proved. Using such a theorem, it is shown that if a curve is a simple closed curve, then it is not horizontally degenerated, neither is it vertically degenerated.

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The articles [15], [18], [1], [17], [19], [4], [12], [6], [13], [2], [10], [3], [7], [8], [9], [11], [16], and [5] provide the notation and terminology for this paper.

1. Intermediate Value Theorems and Bolzano Theorem

For simplicity, we follow the rules: a, b, d, r_1, r_2, s_1, s_2 are real numbers, r, r_3, r_4 are elements of $\mathbb{R}$, p, q are points of $\mathcal{E}^2_{\mathrm{T}}$, and X, Y are non empty topological spaces.

One can prove the following propositions:

(1) For all real numbers a, b, c holds $c \in [a,b]$ iff $a \leq c$ and $c \leq b$.

(4)[1] Let A, B_1, B_2 be subsets of X. Suppose B_1 is open and B_2 is open and B_1 meets A and B_2 meets A and $A \subseteq B_1 \cup B_2$ and B_1 misses B_2. Then A is not connected.

(5) Let f be a continuous map from X into Y and A be a subset of X. If A is connected, then $f^{\circ}A$ is connected.

(6) For all r_1, r_2 such that $r_1 \leq r_2$ holds $\Omega_{[(r_1),r_2]_{\mathrm{T}}}$ is connected.

(7) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every a such that $A = \{r : a < r\}$ holds A is open.

(8) For every subset A of $\mathbb{R}^{\mathbf{1}}$ and for every a such that $A = \{r : a > r\}$ holds A is open.

(9) Let A be a subset of $\mathbb{R}^{\mathbf{1}}$ and given a. Suppose $a \notin A$ and there exist b, d such that $b \in A$ and $d \in A$ and $b < a$ and $a < d$. Then A is not connected.

(10) Let X be a non empty topological space, x_1, x_2 be points of X, a, b, d be real numbers, and f be a continuous map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose X is connected and $f(x_1) = a$ and $f(x_2) = b$ and $a \leq d$ and $d \leq b$. Then there exists a point x_3 of X such that $f(x_3) = d$.

[1] The propositions (2) and (3) have been removed.

(11) Let X be a non empty topological space, x_1, x_2 be points of X, B be a subset of X, a, b, d be real numbers, and f be a continuous map from X into $\mathbb{R}^{\mathbf{1}}$. Suppose B is connected and $f(x_1) = a$ and $f(x_2) = b$ and $a \leq d$ and $d \leq b$ and $x_1 \in B$ and $x_2 \in B$. Then there exists a point x_3 of X such that $x_3 \in B$ and $f(x_3) = d$.

(12) Let given r_1, r_2, a, b. Suppose $r_1 < r_2$. Let f be a continuous map from $[(r_1), r_2]_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ and given d. Suppose $f(r_1) = a$ and $f(r_2) = b$ and $a < d$ and $d < b$. Then there exists r_4 such that $f(r_4) = d$ and $r_1 < r_4$ and $r_4 < r_2$.

(13) Let given r_1, r_2, a, b. Suppose $r_1 < r_2$. Let f be a continuous map from $[(r_1), r_2]_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ and given d. Suppose $f(r_1) = a$ and $f(r_2) = b$ and $a > d$ and $d > b$. Then there exists r_4 such that $f(r_4) = d$ and $r_1 < r_4$ and $r_4 < r_2$.

(14) Let given r_1, r_2, g be a continuous map from $[(r_1), r_2]_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$, and given s_1, s_2. Suppose $r_1 < r_2$ and $s_1 \cdot s_2 < 0$ and $s_1 = g(r_1)$ and $s_2 = g(r_2)$. Then there exists r_3 such that $g(r_3) = 0$ and $r_1 < r_3$ and $r_3 < r_2$.

(15) Let g be a map from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ and given s_1, s_2. Suppose g is continuous and $g(0) \neq g(1)$ and $s_1 = g(0)$ and $s_2 = g(1)$. Then there exists r_3 such that $0 < r_3$ and $r_3 < 1$ and $g(r_3) = \frac{s_1+s_2}{2}$.

2. Simple Closed Curves Are Not Flat

We now state a number of propositions:

(16) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = \mathrm{proj1}$ holds f is continuous.

(17) For every map f from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that $f = \mathrm{proj2}$ holds f is continuous.

(18) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$. Suppose f is continuous. Then there exists a map g from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that g is continuous and for all r, q such that $r \in$ the carrier of $\mathbb{I}$ and $q = f(r)$ holds $q_{\mathbf{1}} = g(r)$.

(19) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$ and f be a map from $\mathbb{I}$ into $(\mathcal{E}^2_{\rm T})\restriction P$. Suppose f is continuous. Then there exists a map g from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that g is continuous and for all r, q such that $r \in$ the carrier of $\mathbb{I}$ and $q = f(r)$ holds $q_{\mathbf{2}} = g(r)$.

(20) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P satisfies conditions of simple closed curve. Then it is not true that there exists r such that for every p such that $p \in P$ holds $p_{\mathbf{2}} = r$.

(21) Let P be a non empty subset of $\mathcal{E}^2_{\rm T}$. Suppose P satisfies conditions of simple closed curve. Then it is not true that there exists r such that for every p such that $p \in P$ holds $p_{\mathbf{1}} = r$.

(22) For every compact non empty subset C of $\mathcal{E}^2_{\rm T}$ such that C is a simple closed curve holds $\text{N-bound}(C) > \text{S-bound}(C)$.

(23) For every compact non empty subset C of $\mathcal{E}^2_{\rm T}$ such that C is a simple closed curve holds $\text{E-bound}(C) > \text{W-bound}(C)$.

(24) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that P is a simple closed curve holds $\mathrm{S}_{\rm min}(P) \neq \mathrm{N}_{\rm max}(P)$.

(25) For every compact non empty subset P of $\mathcal{E}^2_{\rm T}$ such that P is a simple closed curve holds $\mathrm{W}_{\rm min}(P) \neq \mathrm{E}_{\rm max}(P)$.

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