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Completeness of the Lattices of Domains of a Topological Space[1]

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Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *domain* in T provided $\mathrm{Int}\overline{A} \subseteq A \subseteq \overline{\mathrm{Int}A}$ (see [18] and comp. [9]). This notion is a simple generalization of the notions of open and closed domains in T (see [18]). Our main result is concerned with an extension of the following well–known theorem (see e.g. [2], [12], [8]). For a given topological space the Boolean lattices of all its closed domains and all its open domains are complete. It is proved here, using Mizar System, that *the complemented lattice of all domains of a given topological space is complete*, too (comp. [17]).

It is known that both the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains [17]. However, the following two problems remain open.

Problem 1. Let L be a sublattice of the lattice of all domains. Suppose L is complete, is smallest with respect to inclusion, and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all domains ?

A domain in T is said to be a *Borel domain* provided it is a Borel set. Of course every open (closed) domain is a Borel domain. It can be proved that all Borel domains form a sublattice of the lattice of domains.

Problem 2. Let L be a sublattice of the lattice of all domains. Suppose L is smallest with respect to inclusion and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all Borel domains ?

Note that in the beginning the closure and the interior operations for families of subsets of topological spaces are introduced and their important properties are presented (comp. [11], [10], [12]). Using these notions, certain properties of domains, closed domains and open domains are studied (comp. [10], [8]).

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The articles [15], [16], [13], [5], [7], [14], [18], [6], [3], [4], [19], [1], and [17] provide the notation and terminology for this paper.

1. Preliminary Theorems about Subsets of Topological Spaces

In this paper T is a non empty topological space.

We now state several propositions:

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(1) For every subset A of T holds $\mathrm{Int}\,\overline{\mathrm{Int}\,A} \subseteq \mathrm{Int}\,\overline{A}$ and $\mathrm{Int}\,\overline{\mathrm{Int}\,A} \subseteq \overline{\mathrm{Int}\,A}$.

(2) For every subset A of T holds $\overline{\mathrm{Int}\,A} \subseteq \overline{\mathrm{Int}\,\overline{A}}$ and $\mathrm{Int}\,\overline{A} \subseteq \overline{\mathrm{Int}\,\overline{A}}$.

(3) For every subset A of T and for every subset B of T such that B is closed holds if $\overline{\mathrm{Int}(A \cap B)} = A$, then $A \subseteq B$.

(4) For every subset A of T and for every subset B of T such that A is open holds if $\mathrm{Int}\,\overline{A \cup B} = B$, then $A \subseteq B$.

(5) For every subset A of T such that $A \subseteq \overline{\mathrm{Int}\,A}$ holds $A \cup \mathrm{Int}\,\overline{A} \subseteq \overline{\mathrm{Int}(A \cup \mathrm{Int}\,\overline{A})}$.

(6) For every subset A of T such that $\mathrm{Int}\,\overline{A} \subseteq A$ holds $\mathrm{Int}\,\overline{A \cap \overline{\mathrm{Int}\,A}} \subseteq A \cap \overline{\mathrm{Int}\,A}$.

2. The Closure and the Interior Operations for Families of Subsets of a Topological Space

In the sequel T denotes a non empty topological space.

Let us consider T and let F be a family of subsets of T. We introduce $\overline{F}$ as a synonym of $\mathrm{clf}\,F$.

We now state several propositions:

(7) For every family F of subsets of T holds $\overline{F} = \{A;A$ ranges over subsets of T: $\bigvee_{B:\,\text{subset of } T} (A = \overline{B} \wedge B \in F)\}$.

(8) For every family F of subsets of T holds $\overline{F} = \overline{\overline{F}}$.

(9) For every family F of subsets of T holds $F = \emptyset$ iff $\overline{F} = \emptyset$.

(10) For all families F, G of subsets of T holds $\overline{F \cap G} \subseteq \overline{F} \cap \overline{G}$.

(11) For all families F, G of subsets of T holds $\overline{F} \setminus \overline{G} \subseteq \overline{F \setminus G}$.

(12) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\bigcap \overline{F} \subseteq \overline{A}$ and $\overline{A} \subseteq \bigcup \overline{F}$.

(13) For every family F of subsets of T holds $\bigcap F \subseteq \bigcap \overline{F}$.

(14) For every family F of subsets of T holds $\overline{\bigcap F} \subseteq \bigcap \overline{F}$.

(15) For every family F of subsets of T holds $\bigcup \overline{F} \subseteq \overline{\bigcup F}$.

Let us consider T and let F be a family of subsets of T. The functor $\mathrm{Int}\,F$ yields a family of subsets of T and is defined as follows:

(Def. 1) For every subset A of T holds $A \in \mathrm{Int}\,F$ iff there exists a subset B of T such that $A = \mathrm{Int}\,B$ and $B \in F$.

We now state a number of propositions:

(16) For every family F of subsets of T holds $\mathrm{Int}\,F = \{A;A$ ranges over subsets of T: $\bigvee_{B:\,\text{subset of } T} (A = \mathrm{Int}\,B \wedge B \in F)\}$.

(17) For every family F of subsets of T holds $\mathrm{Int}\,F = \mathrm{Int}\,\mathrm{Int}\,F$.

(18) For every family F of subsets of T holds $\mathrm{Int}\,F$ is open.

(19) For every family F of subsets of T holds $F = \emptyset$ iff $\mathrm{Int}\,F = \emptyset$.

(20) For every subset A of T and for every family F of subsets of T such that $F = \{A\}$ holds $\mathrm{Int}\,F = \{\mathrm{Int}\,A\}$.

(21) For all families F, G of subsets of T such that $F \subseteq G$ holds $\mathrm{Int}\,F \subseteq \mathrm{Int}\,G$.

(22) For all families F, G of subsets of T holds $\mathrm{Int}(F \cup G) = \mathrm{Int}F \cup \mathrm{Int}G$.

(23) For all families F, G of subsets of T holds $\mathrm{Int}(F \cap G) \subseteq \mathrm{Int}F \cap \mathrm{Int}G$.

(24) For all families F, G of subsets of T holds $\mathrm{Int}F \setminus \mathrm{Int}G \subseteq \mathrm{Int}(F \setminus G)$.

(25) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\mathrm{Int}A \subseteq \bigcup \mathrm{Int}F$ and $\bigcap \mathrm{Int}F \subseteq \mathrm{Int}A$.

(26) For every family F of subsets of T holds $\bigcup \mathrm{Int}F \subseteq \bigcup F$.

(27) For every family F of subsets of T holds $\bigcap \mathrm{Int}F \subseteq \bigcap F$.

(28) For every family F of subsets of T holds $\bigcup \mathrm{Int}F \subseteq \mathrm{Int}\bigcup F$.

(29) For every family F of subsets of T holds $\mathrm{Int}\bigcap F \subseteq \bigcap \mathrm{Int}F$.

(30) For every family F of subsets of T such that F is finite holds $\mathrm{Int}\bigcap F = \bigcap \mathrm{Int}F$.

In the sequel F is a family of subsets of T.
One can prove the following propositions:

(31) $\overline{\mathrm{Int}F} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\,\text{subset of } T}\ (A = \overline{\mathrm{Int}B} \wedge B \in F)\}$.

(32) $\mathrm{Int}\overline{F} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\,\text{subset of } T}\ (A = \mathrm{Int}\overline{B} \wedge B \in F)\}$.

(33) $\overline{\mathrm{Int}\overline{F}} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\,\text{subset of } T}\ (A = \overline{\mathrm{Int}\overline{B}} \wedge B \in F)\}$.

(34) $\mathrm{Int}\overline{\mathrm{Int}F} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\,\text{subset of } T}\ (A = \mathrm{Int}\overline{\mathrm{Int}B} \wedge B \in F)\}$.

(35) $\overline{\mathrm{Int}\overline{\mathrm{Int}F}} = \overline{\mathrm{Int}F}$.

(36) $\mathrm{Int}\overline{\mathrm{Int}\overline{F}} = \mathrm{Int}\overline{F}$.

(37) $\bigcup \mathrm{Int}\overline{F} \subseteq \bigcup \overline{\mathrm{Int}\overline{F}}$.

(38) $\bigcap \mathrm{Int}\overline{F} \subseteq \bigcap \overline{\mathrm{Int}\overline{F}}$.

(39) $\bigcup \overline{\mathrm{Int}F} \subseteq \bigcup \overline{\mathrm{Int}\overline{F}}$.

(40) $\bigcap \overline{\mathrm{Int}F} \subseteq \bigcap \overline{\mathrm{Int}\overline{F}}$.

(41) $\bigcup \mathrm{Int}\overline{\mathrm{Int}F} \subseteq \bigcup \mathrm{Int}\overline{F}$.

(42) $\bigcap \mathrm{Int}\overline{\mathrm{Int}F} \subseteq \bigcap \mathrm{Int}\overline{F}$.

(43) $\bigcup \mathrm{Int}\overline{\mathrm{Int}F} \subseteq \bigcup \overline{\mathrm{Int}F}$.

(44) $\bigcap \mathrm{Int}\overline{\mathrm{Int}F} \subseteq \bigcap \overline{\mathrm{Int}F}$.

(45) $\bigcup \overline{\mathrm{Int}\overline{F}} \subseteq \bigcup \overline{F}$.

(46) $\bigcap \overline{\mathrm{Int}\overline{F}} \subseteq \bigcap \overline{F}$.

(47) $\bigcup \mathrm{Int}F \subseteq \bigcup \mathrm{Int}\overline{\mathrm{Int}F}$.

(48) $\bigcap \mathrm{Int}F \subseteq \bigcap \mathrm{Int}\overline{\mathrm{Int}F}$.

(49) $\bigcup \overline{\mathrm{Int}F} \subseteq \overline{\mathrm{Int}\bigcup F}$.

(50) $\overline{\mathrm{Int}\bigcap F} \subseteq \bigcap \overline{\mathrm{Int}F}$.

(51) $\bigcup \mathrm{Int}\overline{F} \subseteq \mathrm{Int}\overline{\bigcup F}$.

(52) $\mathrm{Int}\overline{\bigcap F} \subseteq \bigcap \mathrm{Int}\overline{F}$.

(53) $\bigcup \overline{\mathrm{Int}\overline{F}} \subseteq \overline{\mathrm{Int}\overline{\bigcup F}}$.

(54) $\overline{\mathrm{Int}\overline{\bigcap F}} \subseteq \bigcap \overline{\mathrm{Int}\overline{F}}$.

(55) $\bigcup \mathrm{Int}\,\overline{\mathrm{Int}F} \subseteq \mathrm{Int}\,\overline{\mathrm{Int}\bigcup F}$.

(56) $\mathrm{Int}\,\overline{\mathrm{Int}\bigcap F} \subseteq \bigcap \mathrm{Int}\,\overline{\mathrm{Int}F}$.

(57) Let F be a family of subsets of T. Suppose that for every subset A of T such that $A \in F$ holds $A \subseteq \overline{\mathrm{Int}A}$. Then $\bigcup F \subseteq \overline{\mathrm{Int}\bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int}\overline{\bigcup F}}$.

(58) Let F be a family of subsets of T. Suppose that for every subset A of T such that $A \in F$ holds $\mathrm{Int}\overline{A} \subseteq A$. Then $\mathrm{Int}\overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int}\,\overline{\mathrm{Int}\bigcap F} = \mathrm{Int}\bigcap F$.

3. Selected Properties of Domains of a Topological Space

In the sequel T denotes a non empty topological space.

Next we state several propositions:

(59) For all subsets A, B of T such that B is condensed holds $\mathrm{Int}\overline{A \cup B} \cup (A \cup B) = B$ iff $A \subseteq B$.

(60) For all subsets A, B of T such that A is condensed holds $\overline{\mathrm{Int}(A \cap B)} \cap (A \cap B) = A$ iff $A \subseteq B$.

(61) Let A, B be subsets of T. Suppose A is closed condensed and B is closed condensed. Then $\mathrm{Int}A \subseteq \mathrm{Int}B$ if and only if $A \subseteq B$.

(62) For all subsets A, B of T such that A is open condensed and B is open condensed holds $\overline{A} \subseteq \overline{B}$ iff $A \subseteq B$.

(63) For all subsets A, B of T such that A is closed condensed holds if $A \subseteq B$, then $\overline{\mathrm{Int}(A \cap B)} = A$.

(64) For all subsets A, B of T such that B is open condensed holds if $A \subseteq B$, then $\mathrm{Int}\overline{A \cup B} = B$.

Let us consider T and let I_1 be a family of subsets of T. We say that I_1 is domains-family if and only if:

(Def. 2) For every subset A of T such that $A \in I_1$ holds A is condensed.

Next we state several propositions:

(65) For every family F of subsets of T holds $F \subseteq$ the domains of T iff F is domains-family.

(66) For every family F of subsets of T such that F is domains-family holds $\bigcup F \subseteq \overline{\mathrm{Int}\bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int}\overline{\bigcup F}}$.

(67) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int}\overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int}\,\overline{\mathrm{Int}\bigcap F} = \mathrm{Int}\bigcap F$.

(68) For every family F of subsets of T such that F is domains-family holds $\bigcup F \cup \mathrm{Int}\overline{\bigcup F}$ is condensed.

(69) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $B \subseteq \bigcup F \cup \mathrm{Int}\overline{\bigcup F}$, and

(ii) for every subset A of T such that A is condensed holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\bigcup F \cup \mathrm{Int}\overline{\bigcup F} \subseteq A$.

(70) For every family F of subsets of T such that F is domains-family holds $\bigcap F \cap \overline{\mathrm{Int}\bigcap F}$ is condensed.

(71) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $\bigcap F \cap \overline{\mathrm{Int} \bigcap F} \subseteq B$, and

(ii) $F = \emptyset$ or for every subset A of T such that A is condensed holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \bigcap F \cap \overline{\mathrm{Int} \bigcap F}$.

Let us consider T and let I_1 be a family of subsets of T. We say that I_1 is closed-domains-family if and only if:

(Def. 3) For every subset A of T such that $A \in I_1$ holds A is closed condensed.

We now state several propositions:

(72) Let F be a family of subsets of T. Then $F \subseteq$ the closed domains of T if and only if F is closed-domains-family.

(73) For every family F of subsets of T such that F is closed-domains-family holds F is domains-family.

(74) For every family F of subsets of T such that F is closed-domains-family holds F is closed.

(75) For every family F of subsets of T such that F is domains-family holds $\overline{F}$ is closed-domains-family.

(76) Let F be a family of subsets of T. Suppose F is closed-domains-family. Then $\overline{\bigcup F}$ is closed condensed and $\overline{\mathrm{Int} \bigcap F}$ is closed condensed.

(77) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $B \subseteq \overline{\bigcup F}$, and

(ii) for every subset A of T such that A is closed condensed holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\overline{\bigcup F} \subseteq A$.

(78) Let F be a family of subsets of T. Then

(i) if F is closed, then for every subset B of T such that $B \in F$ holds $\overline{\mathrm{Int} \bigcap F} \subseteq B$, and

(ii) $F = \emptyset$ or for every subset A of T such that A is closed condensed holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \overline{\mathrm{Int} \bigcap F}$.

Let us consider T and let I_1 be a family of subsets of T. We say that I_1 is open-domains-family if and only if:

(Def. 4) For every subset A of T such that $A \in I_1$ holds A is open condensed.

We now state several propositions:

(79) For every family F of subsets of T holds $F \subseteq$ the open domains of T iff F is open-domains-family.

(80) For every family F of subsets of T such that F is open-domains-family holds F is domains-family.

(81) For every family F of subsets of T such that F is open-domains-family holds F is open.

(82) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int} F$ is open-domains-family.

(83) Let F be a family of subsets of T. Suppose F is open-domains-family. Then $\mathrm{Int} \bigcap F$ is open condensed and $\mathrm{Int} \overline{\bigcup F}$ is open condensed.

(84) Let F be a family of subsets of T. Then

(i) if F is open, then for every subset B of T such that $B \in F$ holds $B \subseteq \mathrm{Int} \overline{\bigcup F}$, and

(ii) for every subset A of T such that A is open condensed holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\mathrm{Int} \overline{\bigcup F} \subseteq A$.

(85) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $\mathrm{Int} \bigcap F \subseteq B$, and

(ii) $F = \emptyset$ or for every subset A of T such that A is open condensed holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \mathrm{Int} \bigcap F$.

4. Completeness of the Lattice of Domains

In the sequel T is a non empty topological space.

The following propositions are true:

(86) The carrier of the lattice of domains of $T =$ the domains of T.

(87) Let a, b be elements of the lattice of domains of T and A, B be elements of the domains of T. If $a = A$ and $b = B$, then $a \sqcup b = \mathrm{Int}\overline{A \cup B} \cup (A \cup B)$ and $a \sqcap b = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

(88) $\perp_{\text{the lattice of domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of domains of } T} = \Omega_T$.

(89) Let a, b be elements of the lattice of domains of T and A, B be elements of the domains of T. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(90) Let X be a subset of the lattice of domains of T. Then there exists an element a of the lattice of domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(91) The lattice of domains of T is complete.

(92) Let F be a family of subsets of T. Suppose F is domains-family. Let X be a subset of the lattice of domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of domains of } T)} X = \bigcup F \cup \mathrm{Int}\overline{\bigcup F}$.

(93) Let F be a family of subsets of T. Suppose F is domains-family. Let X be a subset of the lattice of domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \bigcap F \cap \overline{\mathrm{Int} \bigcap F}$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

5. Completeness of the Lattices of Closed Domains and Open Domains

In the sequel T is a non empty topological space.

Next we state a number of propositions:

(94) The carrier of the lattice of closed domains of $T =$ the closed domains of T.

(95) Let a, b be elements of the lattice of closed domains of T and A, B be elements of the closed domains of T. If $a = A$ and $b = B$, then $a \sqcup b = A \cup B$ and $a \sqcap b = \overline{\mathrm{Int}(A \cap B)}$.

(96) $\perp_{\text{the lattice of closed domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of closed domains of } T} = \Omega_T$.

(97) Let a, b be elements of the lattice of closed domains of T and A, B be elements of the closed domains of T. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(98) Let X be a subset of the lattice of closed domains of T. Then there exists an element a of the lattice of closed domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of closed domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(99) The lattice of closed domains of T is complete.

(100) Let F be a family of subsets of T. Suppose F is closed-domains-family. Let X be a subset of the lattice of closed domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of closed domains of } T)} X = \overline{\bigcup F}$.

(101) Let F be a family of subsets of T. Suppose F is closed-domains-family. Let X be a subset of the lattice of closed domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \overline{\mathrm{Int} \bigcap F}$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \Omega_T$.

(102) Let F be a family of subsets of T. Suppose F is closed-domains-family. Let X be a subset of the lattice of domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \overline{\mathrm{Int} \bigcap F}$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

(103) The carrier of the lattice of open domains of T = the open domains of T.

(104) Let a, b be elements of the lattice of open domains of T and A, B be elements of the open domains of T. If $a = A$ and $b = B$, then $a \sqcup b = \mathrm{Int}\overline{A \cup B}$ and $a \sqcap b = A \cap B$.

(105) $\perp_{\text{the lattice of open domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of open domains of } T} = \Omega_T$.

(106) Let a, b be elements of the lattice of open domains of T and A, B be elements of the open domains of T. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(107) Let X be a subset of the lattice of open domains of T. Then there exists an element a of the lattice of open domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of open domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(108) The lattice of open domains of T is complete.

(109) Let F be a family of subsets of T. Suppose F is open-domains-family. Let X be a subset of the lattice of open domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of open domains of } T)} X = \mathrm{Int}\overline{\bigcup F}$.

(110) Let F be a family of subsets of T. Suppose F is open-domains-family. Let X be a subset of the lattice of open domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \mathrm{Int} \bigcap F$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \Omega_T$.

(111) Let F be a family of subsets of T. Suppose F is open-domains-family. Let X be a subset of the lattice of domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of domains of } T)} X = \mathrm{Int}\overline{\bigcup F}$.

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REFERENCES

[1] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[2] Garrett Birkhoff. *Lattice Theory*. Providence, Rhode Island, New York, 1967.

[3] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/pcomps_1.html`.

[4] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/binop_1.html`.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[6] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_2.html`.

[7] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[8] Ryszard Engelking. *General Topology*, volume 60 of *Monografie Matematyczne*. PWN - Polish Scientific Publishers, Warsaw, 1977.

[9] Yoshinori Isomichi. New concepts in the theory of topological space – supercondensed set, subcondensed set, and condensed set. *Pacific Journal of Mathematics*, 38(3):657–668, 1971.

[10] Kazimierz Kuratowski. Sur l'opération $\overline{A}$ de l'analysis situs. *Fundamenta Mathematicae*, 3:182–199, 1922.

[11] Kazimierz Kuratowski. *Topology*, volume I. PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[12] Kazimierz Kuratowski and Andrzej Mostowski. *Set Theory (with an introduction to descriptive set theory)*, volume 86 of *Studies in Logic and The Foundations of Mathematics*. PWN - Polish Scientific Publishers and North-Holland Publishing Company, Warsaw-Amsterdam, 1976.

[13] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[14] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/pre_topc.html`.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[17] Toshihiko Watanabe. The lattice of domains of a topological space. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/tdlat_1.html`.

[18] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_1.html`.

[19] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

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