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# Series of Positive Real Numbers. Measure Theory

Józef Białas
University of Łódź

**Summary.** We introduce properties of a series of nonnegative $\overline{\mathbb{R}}$ numbers, where $\overline{\mathbb{R}}$ denotes the enlarged set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The paper contains definition of sup $F$ and inf $F$, for $F$ being function, and a definition of a sumable subset of $\overline{\mathbb{R}}$. We proved the basic theorems regarding the definitions mentioned above. The work is the second part of a series of articles concerning the Lebesgue measure theory.

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The articles [7], [9], [8], [6], [3], [10], [4], [5], [1], and [2] provide the notation and terminology for this paper.

The extended real number $0_{\overline{\mathbb{R}}}$ is defined by:

(Def. 1) $0_{\overline{\mathbb{R}}} = 0$.

Let $x$, $y$ be extended real numbers. The functor $x + y$ yielding an extended real number is defined as follows:

(Def. 2)(i) There exist real numbers $a$, $b$ such that $x = a$ and $y = b$ and $x + y = a + b$ if $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

(ii) $x + y = +\infty$ if $x = +\infty$ and $y \neq -\infty$ or $y = +\infty$ and $x \neq -\infty$,

(iii) $x + y = -\infty$ if $x = -\infty$ and $y \neq +\infty$ or $y = -\infty$ and $x \neq +\infty$,

(iv) $x + y = 0_{\overline{\mathbb{R}}}$, otherwise.

Let us observe that the functor $x + y$ is commutative.

Next we state two propositions:

(1) For all extended real numbers $x$, $y$ and for all real numbers $a$, $b$ such that $x = a$ and $y = b$ holds $x + y = a + b$.

(2) For every extended real number $x$ holds $x \in \mathbb{R}$ or $x = +\infty$ or $x = -\infty$.

Let $x$ be an extended real number. The functor $-x$ yielding an extended real number is defined by:

(Def. 3)(i) There exists a real number $a$ such that $x = a$ and $-x = -a$ if $x \in \mathbb{R}$,

(ii) $-x = -\infty$ if $x = +\infty$,

(iii) $-x = +\infty$, otherwise.

Let us observe that the functor $-x$ is involutive.

Let $x$, $y$ be extended real numbers. The functor $x-y$ yielding an extended real number is defined as follows:

(Def. 4) $x-y=x+-y$.

We now state a number of propositions:

(3) For every extended real number $x$ and for every real number $a$ such that $x=a$ holds $-x=-a$.

(4) $--\infty=+\infty$.

(5) For all extended real numbers $x$, $y$ and for all real numbers $a$, $b$ such that $x=a$ and $y=b$ holds $x-y=a-b$.

(6) For every extended real number $x$ such that $x\neq+\infty$ holds $+\infty-x=+\infty$ and $x-+\infty=-\infty$.

(7) For every extended real number $x$ such that $x\neq-\infty$ holds $-\infty-x=-\infty$ and $x--\infty=+\infty$.

(8) For all extended real numbers $x$, $s$ such that $x+s=+\infty$ holds $x=+\infty$ or $s=+\infty$.

(9) For all extended real numbers $x$, $s$ such that $x+s=-\infty$ holds $x=-\infty$ or $s=-\infty$.

(10) For all extended real numbers $x$, $s$ such that $x-s=+\infty$ holds $x=+\infty$ or $s=-\infty$.

(11) For all extended real numbers $x$, $s$ such that $x-s=-\infty$ holds $x=-\infty$ or $s=+\infty$.

(12) For all extended real numbers $x$, $s$ such that $x\neq+\infty$ or $s\neq-\infty$ but $x\neq-\infty$ or $s\neq+\infty$ and $x+s\in\mathbb{R}$ holds $x\in\mathbb{R}$ and $s\in\mathbb{R}$.

(13) For all extended real numbers $x$, $s$ such that $x\neq+\infty$ or $s\neq+\infty$ but $x\neq-\infty$ or $s\neq-\infty$ and $x-s\in\mathbb{R}$ holds $x\in\mathbb{R}$ and $s\in\mathbb{R}$.

(14) Let $x$, $y$, $s$, $t$ be extended real numbers. Suppose $x\neq+\infty$ or $s\neq-\infty$ but $x\neq-\infty$ or $s\neq+\infty$ and $y\neq+\infty$ or $t\neq-\infty$ but $y\neq-\infty$ or $t\neq+\infty$ and $x\leq y$ and $s\leq t$. Then $x+s\leq y+t$.

(15) Let $x$, $y$, $s$, $t$ be extended real numbers. Suppose $x\neq+\infty$ or $t\neq+\infty$ but $x\neq-\infty$ or $t\neq-\infty$ and $y\neq+\infty$ or $s\neq+\infty$ but $y\neq-\infty$ or $s\neq-\infty$ and $x\leq y$ and $s\leq t$. Then $x-t\leq y-s$.

(16) For all extended real numbers $x$, $y$ holds $x\leq y$ iff $-y\leq-x$.

(17) For all extended real numbers $x$, $y$ holds $x<y$ iff $-y<-x$.

(18) For every extended real number $x$ holds $x+0_{\overline{\mathbb{R}}}=x$ and $0_{\overline{\mathbb{R}}}+x=x$.

(19) $-\infty<0_{\overline{\mathbb{R}}}$ and $0_{\overline{\mathbb{R}}}<+\infty$.

(20) For all extended real numbers $x$, $y$, $z$ such that $0_{\overline{\mathbb{R}}}\leq z$ and $0_{\overline{\mathbb{R}}}\leq x$ and $y=x+z$ holds $x\leq y$.

(21) For every extended real number $x$ such that $x\in\mathbb{N}$ holds $0_{\overline{\mathbb{R}}}\leq x$.

Let $X$, $Y$ be non empty subsets of $\overline{\mathbb{R}}$. The functor $X+Y$ yields a subset of $\overline{\mathbb{R}}$ and is defined by the condition (Def. 5).

(Def. 5) Let $z$ be an extended real number. Then $z\in X+Y$ if and only if there exist extended real numbers $x$, $y$ such that $x\in X$ and $y\in Y$ and $z=x+y$.

Let $X$, $Y$ be non empty subsets of $\overline{\mathbb{R}}$. Observe that $X+Y$ is non empty.

Let $X$ be a non empty subset of $\overline{\mathbb{R}}$. The functor $-X$ yielding a subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 6) For every extended real number $z$ holds $z\in-X$ iff there exists an extended real number $x$ such that $x\in X$ and $z=-x$.

Let $X$ be a non empty subset of $\overline{\mathbb{R}}$. Note that $-X$ is non empty.

One can prove the following propositions:

(22) For every non empty subset $X$ of $\overline{\mathbb{R}}$ holds $--X = X$.

(23) For every non empty subset $X$ of $\overline{\mathbb{R}}$ and for every extended real number $z$ holds $z \in X$ iff $-z \in -X$.

(24) For all non empty subsets $X$, $Y$ of $\overline{\mathbb{R}}$ holds $X \subseteq Y$ iff $-X \subseteq -Y$.

(25) For every extended real number $z$ holds $z \in \mathbb{R}$ iff $-z \in \mathbb{R}$.

(26) Let $X$, $Y$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $-\infty \notin X$ or $+\infty \notin Y$ but $+\infty \notin X$ or $-\infty \notin Y$ and $\sup X \neq +\infty$ or $\sup Y \neq -\infty$ but $\sup X \neq -\infty$ or $\sup Y \neq +\infty$. Then $\sup(X+Y) \leq \sup X + \sup Y$.

(27) Let $X$, $Y$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $-\infty \notin X$ or $+\infty \notin Y$ but $+\infty \notin X$ or $-\infty \notin Y$ and $\inf X \neq +\infty$ or $\inf Y \neq -\infty$ but $\inf X \neq -\infty$ or $\inf Y \neq +\infty$. Then $\inf X + \inf Y \leq \inf(X+Y)$.

(28) For all non empty subsets $X$, $Y$ of $\overline{\mathbb{R}}$ such that $X$ is upper bounded and $Y$ is upper bounded holds $\sup(X+Y) \leq \sup X + \sup Y$.

(29) For all non empty subsets $X$, $Y$ of $\overline{\mathbb{R}}$ such that $X$ is lower bounded and $Y$ is lower bounded holds $\inf X + \inf Y \leq \inf(X+Y)$.

(30) Let $X$ be a non empty subset of $\overline{\mathbb{R}}$ and $a$ be an extended real number. Then $a$ is a majorant of $X$ if and only if $-a$ is a minorant of $-X$.

(31) Let $X$ be a non empty subset of $\overline{\mathbb{R}}$ and $a$ be an extended real number. Then $a$ is a minorant of $X$ if and only if $-a$ is a majorant of $-X$.

(32) For every non empty subset $X$ of $\overline{\mathbb{R}}$ holds $\inf(-X) = -\sup X$.

(33) For every non empty subset $X$ of $\overline{\mathbb{R}}$ holds $\sup(-X) = -\inf X$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. Then $\operatorname{rng} F$ is a non empty subset of $\overline{\mathbb{R}}$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. The functor $\sup F$ yielding an extended real number is defined by:

(Def. 7) $\sup F = \sup \operatorname{rng} F$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. The functor $\inf F$ yielding an extended real number is defined as follows:

(Def. 8) $\inf F = \inf \operatorname{rng} F$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, let $F$ be a function from $X$ into $Y$, and let $x$ be an element of $X$. Then $F(x)$ is an extended real number.

The scheme *FunctR ealEx* deals with a non empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a function $f$ from $\mathcal{A}$ into $\mathcal{B}$ such that for every element $x$ of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

provided the parameters have the following property:

- For every element $x$ of $\mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{B}$.

Let $X$ be a non empty set, let $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$, let $F$ be a function from $X$ into $Y$, and let $G$ be a function from $X$ into $Z$. The functor $F+G$ yielding a function from $X$ into $Y+Z$ is defined by:

(Def. 9) For every element $x$ of $X$ holds $(F+G)(x) = F(x)+G(x)$.

One can prove the following three propositions:

(34) Let $X$ be a non empty set, $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$, $F$ be a function from $X$ into $Y$, and $G$ be a function from $X$ into $Z$. Then $\operatorname{rng}(F+G) \subseteq \operatorname{rng} F + \operatorname{rng} G$.

(35) Let $X$ be a non empty set and $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $-\infty \notin Y$ or $+\infty \notin Z$ but $+\infty \notin Y$ or $-\infty \notin Z$. Let $F$ be a function from $X$ into $Y$ and $G$ be a function from $X$ into $Z$. If $\sup F \neq +\infty$ or $\sup G \neq -\infty$ and if $\sup F \neq -\infty$ or $\sup G \neq +\infty$, then $\sup(F+G) \leq \sup F + \sup G$.

(36) Let $X$ be a non empty set and $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $-\infty \notin Y$ or $+\infty \notin Z$ but $+\infty \notin Y$ or $-\infty \notin Z$. Let $F$ be a function from $X$ into $Y$ and $G$ be a function from $X$ into $Z$. If $\inf F \neq +\infty$ or $\inf G \neq -\infty$ and if $\inf F \neq -\infty$ or $\inf G \neq +\infty$, then $\inf F + \inf G \leq \inf(F+G)$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. The functor $-F$ yields a function from $X$ into $-Y$ and is defined by:

(Def. 10) For every element $x$ of $X$ holds $(-F)(x) = -F(x)$.

The following propositions are true:

(37) For every non empty set $X$ and for every non empty subset $Y$ of $\overline{\mathbb{R}}$ and for every function $F$ from $X$ into $Y$ holds $\operatorname{rng}(-F) = -\operatorname{rng} F$.

(38) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. Then $\inf(-F) = -\sup F$ and $\sup(-F) = -\inf F$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. We say that $F$ is upper bounded if and only if:

(Def. 11) $\sup F < +\infty$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. We say that $F$ is lower bounded if and only if:

(Def. 12) $-\infty < \inf F$.

Let $X$ be a non empty set, let $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and let $F$ be a function from $X$ into $Y$. We say that $F$ is bounded if and only if:

(Def. 13) $F$ is upper bounded and lower bounded.

Let $X$ be a non empty set and let $Y$ be a non empty subset of $\overline{\mathbb{R}}$. Observe that every function from $X$ into $Y$ which is bounded is also upper bounded and lower bounded and every function from $X$ into $Y$ which is upper bounded and lower bounded is also bounded.

We now state a number of propositions:

(39) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. Then $F$ is bounded if and only if $\sup F < +\infty$ and $-\infty < \inf F$.

(40) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. Then $F$ is upper bounded if and only if $-F$ is lower bounded.

(41) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. Then $F$ is lower bounded if and only if $-F$ is upper bounded.

(42) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. Then $F$ is bounded if and only if $-F$ is bounded.

(43) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, $F$ be a function from $X$ into $Y$, and $x$ be an element of $X$. Then $-\infty \leq F(x)$ and $F(x) \leq +\infty$.

(44) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, $F$ be a function from $X$ into $Y$, and $x$ be an element of $X$. If $Y \subseteq \mathbb{R}$, then $-\infty < F(x)$ and $F(x) < +\infty$.

(45) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, $F$ be a function from $X$ into $Y$, and $x$ be an element of $X$. Then $\inf F \leq F(x)$ and $F(x) \leq \sup F$.

(46) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. If $Y \subseteq \mathbb{R}$, then $F$ is upper bounded iff $\sup F \in \mathbb{R}$.

(47) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. If $Y \subseteq \mathbb{R}$, then $F$ is lower bounded iff $\inf F \in \mathbb{R}$.

(48) Let $X$ be a non empty set, $Y$ be a non empty subset of $\overline{\mathbb{R}}$, and $F$ be a function from $X$ into $Y$. If $Y \subseteq \mathbb{R}$, then $F$ is bounded iff $\inf F \in \mathbb{R}$ and $\sup F \in \mathbb{R}$.

(49) Let $X$ be a non empty set and $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$. Let $F_1$ be a function from $X$ into $Y$ and $F_2$ be a function from $X$ into $Z$. If $F_1$ is upper bounded and $F_2$ is upper bounded, then $F_1 + F_2$ is upper bounded.

(50) Let $X$ be a non empty set and $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$. Let $F_1$ be a function from $X$ into $Y$ and $F_2$ be a function from $X$ into $Z$. If $F_1$ is lower bounded and $F_2$ is lower bounded, then $F_1 + F_2$ is lower bounded.

(51) Let $X$ be a non empty set and $Y$, $Z$ be non empty subsets of $\overline{\mathbb{R}}$. Suppose $Y \subseteq \mathbb{R}$ and $Z \subseteq \mathbb{R}$. Let $F_1$ be a function from $X$ into $Y$ and $F_2$ be a function from $X$ into $Z$. If $F_1$ is bounded and $F_2$ is bounded, then $F_1 + F_2$ is bounded.

(52) There exists a function $F$ from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that $F$ is one-to-one and $\mathbb{N} = \operatorname{rng} F$ and $\operatorname{rng} F$ is a non empty subset of $\overline{\mathbb{R}}$.

Let $D$ be a non empty set and let $I_1$ be a subset of $D$. Let us observe that $I_1$ is countable if and only if:

(Def. 14) $I_1$ is empty or there exists a function $F$ from $\mathbb{N}$ into $D$ such that $I_1 = \operatorname{rng} F$.

We introduce $I_1$ is denumerable as a synonym of $I_1$ is countable.

Let us observe that there exists a non empty subset of $\overline{\mathbb{R}}$ which is denumerable.

A denumerable set of larged real is a denumerable non empty subset of $\overline{\mathbb{R}}$.

Let $I_1$ be a set. We say that $I_1$ is non-negative if and only if:

(Def. 15) For every extended real number $x$ such that $x \in I_1$ holds $0_{\overline{\mathbb{R}}} \leq x$.

Let us note that there exists a denumerable set of larged real which is non-negative.

A denumerable set of positive larged real is a non-negative denumerable set of larged real.

Let $D$ be a denumerable set of larged real. A function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ is said to be a numeration of $D$ if:

(Def. 16) $D = \operatorname{rng} \mathrm{it}$.

Let $N$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and let $n$ be a natural number. Then $N(n)$ is an extended real number.

One can prove the following proposition

(53) Let $D$ be a denumerable set of larged real and $N$ be a numeration of $D$. Then there exists a function $F$ from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that

(i) $F(0) = N(0)$, and

(ii) for every natural number $n$ and for every extended real number $y$ such that $y = F(n)$ holds $F(n+1) = y + N(n+1)$.

Let $D$ be a denumerable set of larged real and let $N$ be a numeration of $D$. The functor $\operatorname{Ser}(D,N)$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by the conditions (Def. 17).

(Def. 17)(i) $\operatorname{Ser}(D,N)(0) = N(0)$, and

(ii) for every natural number $n$ and for every extended real number $y$ such that $y = \operatorname{Ser}(D,N)(n)$ holds $\operatorname{Ser}(D,N)(n+1) = y + N(n+1)$.

The following three propositions are true:

(54) Let $D$ be a denumerable set of positive larged real, $N$ be a numeration of $D$, and $n$ be a natural number. Then $0_{\overline{\mathbb{R}}} \leq N(n)$.

(55) Let $D$ be a denumerable set of positive larged real, $N$ be a numeration of $D$, and $n$ be a natural number. Then $\mathrm{Ser}(D,N)(n) \leq \mathrm{Ser}(D,N)(n+1)$ and $0_{\overline{\mathbb{R}}} \leq \mathrm{Ser}(D,N)(n)$.

(56) Let $D$ be a denumerable set of positive larged real, $N$ be a numeration of $D$, and $n$, $m$ be natural numbers. Then $\mathrm{Ser}(D,N)(n) \leq \mathrm{Ser}(D,N)(n+m)$.

Let $D$ be a denumerable set of larged real. A non empty subset of $\overline{\mathbb{R}}$ is said to be a set of series of $D$ if:

(Def. 18) There exists a numeration $N$ of $D$ such that it $= \mathrm{rng}\,\mathrm{Ser}(D,N)$.

Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\mathrm{rng}\,F$ is a non empty subset of $\overline{\mathbb{R}}$.

Let $D$ be a denumerable set of positive larged real and let $N$ be a numeration of $D$. The functor $\sum_D N$ yielding an extended real number is defined as follows:

(Def. 19) $\sum_D N = \sup \mathrm{rng}\,\mathrm{Ser}(D,N)$.

Let $D$ be a denumerable set of positive larged real and let $N$ be a numeration of $D$. We say that $D$ is $N$ summable if and only if:

(Def. 20) $\sum_D N \in \mathbb{R}$.

Next we state the proposition

(57) For every function $F$ from $\mathbb{N}$ into $\overline{\mathbb{R}}$ holds $\mathrm{rng}\,F$ is a denumerable set of larged real.

Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\mathrm{rng}\,F$ is a denumerable set of larged real.

Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $\mathrm{Ser}\,F$ yields a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and is defined by:

(Def. 21) For every numeration $N$ of $\mathrm{rng}\,F$ such that $N = F$ holds $\mathrm{Ser}\,F = \mathrm{Ser}(\mathrm{rng}\,F, N)$.

Let $X$ be a set and let $F$ be a function from $X$ into $\overline{\mathbb{R}}$. We say that $F$ is non-negative if and only if:

(Def. 22) $\mathrm{rng}\,F$ is non-negative.

Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $\sum F$ yields an extended real number and is defined as follows:

(Def. 23) $\sum F = \sup \mathrm{rng}\,\mathrm{Ser}\,F$.

The following propositions are true:

(58) Let $X$ be a non empty set and $F$ be a function from $X$ into $\overline{\mathbb{R}}$. Then $F$ is non-negative if and only if for every element $n$ of $X$ holds $0_{\overline{\mathbb{R}}} \leq F(n)$.

(59) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ and $n$ be a natural number. If $F$ is non-negative, then $(\mathrm{Ser}\,F)(n) \leq (\mathrm{Ser}\,F)(n+1)$ and $0_{\overline{\mathbb{R}}} \leq (\mathrm{Ser}\,F)(n)$.

(60) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative. Let $n$, $m$ be natural numbers. Then $(\mathrm{Ser}\,F)(n) \leq (\mathrm{Ser}\,F)(n+m)$.

(61) Let $F_1$, $F_2$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F_1$ is non-negative. Suppose that for every natural number $n$ holds $F_1(n) \leq F_2(n)$. Let $n$ be a natural number. Then $(\mathrm{Ser}\,F_1)(n) \leq (\mathrm{Ser}\,F_2)(n)$.

(62) Let $F_1$, $F_2$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F_1$ is non-negative. If for every natural number $n$ holds $F_1(n) \leq F_2(n)$, then $\sum F_1 \leq \sum F_2$.

(63) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) $(\mathrm{Ser}F)(0) = F(0)$, and

(ii) for every natural number $n$ and for every extended real number $y$ such that $y = (\mathrm{Ser}F)(n)$ holds $(\mathrm{Ser}F)(n+1) = y + F(n+1)$.

(64) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative. If there exists a natural number $n$ such that $F(n) = +\infty$, then $\sum F = +\infty$.

Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. We say that $F$ is summable if and only if:

(Def. 24) $\sum F \in \mathbb{R}$.

The following propositions are true:

(65) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative. If there exists a natural number $n$ such that $F(n) = +\infty$, then $F$ is not summable.

(66) Let $F_1$, $F_2$ be functions from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F_1$ is non-negative. Suppose that for every natural number $n$ holds $F_1(n) \leq F_2(n)$. If $F_2$ is summable, then $F_1$ is summable.

(67) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative. Let $n$ be a natural number. If for every natural number $r$ such that $n \leq r$ holds $F(r) = 0_{\overline{\mathbb{R}}}$, then $\sum F = (\mathrm{Ser}F)(n)$.

(68) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose that for every natural number $n$ holds $F(n) \in \mathbb{R}$. Let $n$ be a natural number. Then $(\mathrm{Ser}F)(n) \in \mathbb{R}$.

(69) Let $F$ be a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $F$ is non-negative. Given a natural number $n$ such that for every natural number $k$ such that $n \leq k$ holds $F(k) = 0_{\overline{\mathbb{R}}}$ and for every natural number $k$ such that $k \leq n$ holds $F(k) \neq +\infty$. Then $F$ is summable.

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