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Convergent Real Sequences. Upper and Lower Bound of Sets of Real Numbers

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Summary. The article contains theorems about convergent sequences and the limit of sequences occurring in [5] such as Bolzano-Weierstrass theorem, Cauchy theorem and others. Bounded sets of real numbers and lower and upper bound of subset of real numbers are defined.

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The articles [9], [12], [2], [11], [4], [13], [7], [5], [1], [3], [6], [8], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: n, k, m denote natural numbers, r, r_1, p, g, g_1, g_2, s denote real numbers, s_1, s_2 denote sequences of real numbers, N_1 denotes an increasing sequence of naturals, and X, Y denote subsets of $\mathbb{R}$.

Next we state several propositions:

(1) If $0 < r_1$ and $r_1 \leq r$ and $0 \leq g$, then $\frac{g}{r} \leq \frac{g}{r_1}$.

(4)[1] If $0 < s$, then $0 < \frac{s}{3}$.

(6)[2] If $0 < g$ and $0 \leq r$ and $g \leq g_1$ and $r < r_1$, then $g \cdot r < g_1 \cdot r_1$.

(7) If $0 \leq g$ and $0 \leq r$ and $g \leq g_1$ and $r \leq r_1$, then $g \cdot r \leq g_1 \cdot r_1$.

(8) Let given X, Y. Suppose that for all r, p such that $r \in X$ and $p \in Y$ holds $r < p$. Then there exists g such that for all r, p such that $r \in X$ and $p \in Y$ holds $r \leq g$ and $g \leq p$.

(9) If $0 < p$ and there exists r such that $r \in X$ and for every r such that $r \in X$ holds $r + p \in X$, then for every g there exists r such that $r \in X$ and $g < r$.

(10) For every r there exists n such that $r < n$.

Let X be a real-membered set. We say that X is upper bounded if and only if:

(Def. 1) There exists p such that for every r such that $r \in X$ holds $r \leq p$.

We say that X is lower bounded if and only if:

(Def. 2) There exists p such that for every r such that $r \in X$ holds $p \leq r$.

[1] The propositions (2) and (3) have been removed.
[2] The proposition (5) has been removed.

Let us consider X. We say that X is bounded if and only if:

(Def. 3) X is lower bounded and upper bounded.

One can prove the following proposition

(14)[3] X is bounded iff there exists s such that $0 < s$ and for every r such that $r \in X$ holds $|r| < s$.

Let us consider r. Then $\{r\}$ is a subset of $\mathbb{R}$.
The following propositions are true:

(15) $\{r\}$ is bounded.

(16) Let X be a real-membered set. Suppose X is non empty and upper bounded. Then there exists g such that for every r such that $r \in X$ holds $r \leq g$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $g - s < r$.

(17) Let X be a real-membered set. Suppose that

(i) for every r such that $r \in X$ holds $r \leq g_1$,

(ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_1 - s < r$,

(iii) for every r such that $r \in X$ holds $r \leq g_2$, and

(iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $g_2 - s < r$.

Then $g_1 = g_2$.

(18) Let X be a real-membered set. Suppose X is non empty and lower bounded. Then there exists g such that for every r such that $r \in X$ holds $g \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g + s$.

(19) Let X be a real-membered set. Suppose that

(i) for every r such that $r \in X$ holds $g_1 \leq r$,

(ii) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_1 + s$,

(iii) for every r such that $r \in X$ holds $g_2 \leq r$, and

(iv) for every s such that $0 < s$ there exists r such that $r \in X$ and $r < g_2 + s$.

Then $g_1 = g_2$.

Let X be a real-membered set. Let us assume that X is non empty and upper bounded. The functor $\sup X$ yields a real number and is defined as follows:

(Def. 4) For every r such that $r \in X$ holds $r \leq \sup X$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $\sup X - s < r$.

Let X be a real-membered set. Let us assume that X is non empty and lower bounded. The functor $\inf X$ yielding a real number is defined by:

(Def. 5) For every r such that $r \in X$ holds $\inf X \leq r$ and for every s such that $0 < s$ there exists r such that $r \in X$ and $r < \inf X + s$.

Let us consider X. Then $\sup X$ is a real number. Then $\inf X$ is a real number.
One can prove the following propositions:

(22)[4] $\inf\{r\} = r$ and $\sup\{r\} = r$.

(23) $\inf\{r\} = \sup\{r\}$.

(24) If X is bounded and non empty, then $\inf X \leq \sup X$.

[3] The propositions (11)–(13) have been removed.
[4] The propositions (20) and (21) have been removed.

(25) If X is bounded and non empty, then there exist r, p such that $r \in X$ and $p \in X$ and $p \neq r$ iff $\inf X < \sup X$.

(26) If s_1 is convergent, then $|s_1|$ is convergent.

(27) If s_1 is convergent, then $\lim |s_1| = |\lim s_1|$.

(28) If $|s_1|$ is convergent and $\lim |s_1| = 0$, then s_1 is convergent and $\lim s_1 = 0$.

(29) If s_2 is a subsequence of s_1 and s_1 is convergent, then s_2 is convergent.

(30) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(31) If s_1 is convergent and there exists k such that for every n such that $k \leq n$ holds $s_2(n) = s_1(n)$, then s_2 is convergent.

(32) If s_1 is convergent and there exists k such that for every n such that $k \leq n$ holds $s_2(n) = s_1(n)$, then $\lim s_1 = \lim s_2$.

(33) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(35)[5] If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(36) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then $\lim s_2 = \lim s_1$.

(37) If s_1 is convergent and $\lim s_1 \neq 0$, then there exists k such that $s_1 \uparrow k$ is non-zero.

(38) If s_1 is convergent and $\lim s_1 \neq 0$, then there exists s_2 which is a subsequence of s_1 and non-zero.

(39) If s_1 is constant, then s_1 is convergent.

(40) If s_1 is constant and $r \in \operatorname{rng} s_1$ or s_1 is constant and there exists n such that $s_1(n) = r$, then $\lim s_1 = r$.

(41) If s_1 is constant, then for every n holds $\lim s_1 = s_1(n)$.

(42) If s_1 is convergent and $\lim s_1 \neq 0$, then for every s_2 such that s_2 is a subsequence of s_1 and non-zero holds $\lim(s_2{}^{-1}) = (\lim s_1)^{-1}$.

(43) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then s_1 is convergent.

(44) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then $\lim s_1 = 0$.

(45) If for every n holds $s_1(n) = \frac{1}{n+1}$, then s_1 is convergent and $\lim s_1 = 0$.

(46) If $0 < r$ and for every n holds $s_1(n) = \frac{g}{n+r}$, then s_1 is convergent and $\lim s_1 = 0$.

(47) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n \cdot n + r}$, then s_1 is convergent.

(48) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n \cdot n + r}$, then $\lim s_1 = 0$.

(49) If for every n holds $s_1(n) = \frac{1}{n \cdot n + 1}$, then s_1 is convergent and $\lim s_1 = 0$.

(50) If $0 < r$ and for every n holds $s_1(n) = \frac{g}{n \cdot n + r}$, then s_1 is convergent and $\lim s_1 = 0$.

(51) If s_1 is non-decreasing and upper bounded, then s_1 is convergent.

(52) If s_1 is non-increasing and lower bounded, then s_1 is convergent.

(53) If s_1 is monotone and bounded, then s_1 is convergent.

(54) If s_1 is upper bounded and non-decreasing, then for every n holds $s_1(n) \leq \lim s_1$.

[5] The proposition (34) has been removed.

(55) If s_1 is lower bounded and non-increasing, then for every n holds $\lim s_1 \leq s_1(n)$.

(56) For every s_1 there exists N_1 such that $s_1 \cdot N_1$ is monotone.

(57) If s_1 is bounded, then there exists s_2 which is a subsequence of s_1 and convergent.

(58) s_1 is convergent iff for every s such that $0 < s$ there exists n such that for every m such that $n \leq m$ holds $|s_1(m) - s_1(n)| < s$.

(59) If s_1 is constant and s_2 is convergent, then $\lim(s_1 + s_2) = s_1(0) + \lim s_2$ and $\lim(s_1 - s_2) = s_1(0) - \lim s_2$ and $\lim(s_2 - s_1) = \lim s_2 - s_1(0)$ and $\lim(s_1\, s_2) = s_1(0) \cdot \lim s_2$.

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