JOURNAL OF FORMALIZED MATHEMATICS
Volume 9, Released 1997, Published 2003
Inst. of Computer Science, Univ. of Białystok

Bounding Boxes for Compact Sets in $\mathcal{E}^2$

Czesław Byliński
Warsaw University
Białystok

Piotr Rudnicki
University of Alberta
Edmonton

Summary. We define pseudocompact topological spaces and prove that every compact space is pseudocompact. We also solve an exercise from [14] p.225 that for a topological space X the following are equivalent:

- Every continuous real map from X is bounded (i.e. X is pseudocompact).
- Every continuous real map from X attains minimum.
- Every continuous real map from X attains maximum.

Finally, for a compact set in E^2 we define its bounding rectangle and introduce a collection of notions associated with the box.

MML Identifier: PSCOMP_1.

WWW: http://mizar.org/JFM/Vol9/pscomp_1.html

The articles [20], [23], [1], [22], [16], [7], [18], [10], [21], [24], [3], [4], [13], [12], [15], [11], [19], [17], [6], [5], [2], [8], and [9] provide the notation and terminology for this paper.

1. Preliminaries

Let X be a set. Let us observe that X has non empty elements if and only if:

(Def. 1) $0 \notin X$.

We introduce X is without zero as a synonym of X has non empty elements. We introduce X has zero as an antonym of X has non empty elements.

Let us mention that $\mathbb{R}$ has zero and $\mathbb{N}$ has zero.

Let us observe that there exists a set which is non empty and without zero and there exists a set which is non empty and has zero.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty and without zero and there exists a subset of $\mathbb{R}$ which is non empty and has zero.

We now state the proposition

(1) For every set F such that F is non empty and $\subseteq$-linear and has non empty elements holds F is centered.

Let F be a set. One can check that every family of subsets of F which is non empty and $\subseteq$-linear and has non empty elements is also centered.

Let A, B be sets and let f be a function from A into B. Then rng f is a subset of B.

Let X, Y be non empty sets and let f be a function from X into Y. Note that $f^\circ X$ is non empty.

Let X, Y be sets and let f be a function from X into Y. The functor ^{-1}f yields a function from 2^Y into 2^X and is defined by:

(Def. 2) For every subset y of Y holds $(^{-1}f)(y) = f^{-1}(y)$.

We now state the proposition

(2) Let X, Y, x be sets, S be a subset of 2^Y, and f be a function from X into Y. If $x \in \bigcap((^{-1}f)^{\circ}S)$, then $f(x) \in \bigcap S$.

In the sequel r, s, t are real numbers.
One can prove the following propositions:

(3) If $|r|+|s| = 0$, then $r = 0$.

(4) If $r < s$ and $s < t$, then $|s| < |r|+|t|$.

(5) If $-s < r$ and $r < s$, then $|r| < s$.

In the sequel s_1 is a sequence of real numbers and X, Y are subsets of $\mathbb{R}$.
We now state two propositions:

(6) If s_1 is convergent and non-zero and $\lim s_1 = 0$, then ${s_1}^{-1}$ is non bounded.

(7) $\operatorname{rng} s_1$ is bounded iff s_1 is bounded.

Let X be a real-membered set. We introduce $\sup X$ as a synonym of $\sup X$. We introduce $\inf X$ as a synonym of $\inf X$.
Let X be a subset of $\mathbb{R}$. Then $\sup X$ is an element of $\mathbb{R}$. Then $\inf X$ is an element of $\mathbb{R}$.
We now state several propositions:

(8) For every non empty real-membered set X and for every t such that for every s such that $s \in X$ holds $s \geq t$ holds $\inf X \geq t$.

(9) Let X be a non empty real-membered set. Suppose for every s such that $s \in X$ holds $s \geq r$ and for every t such that for every s such that $s \in X$ holds $s \geq t$ holds $r \geq t$. Then $r = \inf X$.

(10) For every non empty real-membered set X and for every r and for every t such that for every s such that $s \in X$ holds $s \leq t$ holds $\sup X \leq t$.

(11) Let X be a non empty real-membered set and given r. Suppose for every s such that $s \in X$ holds $s \leq r$ and for every t such that for every s such that $s \in X$ holds $s \leq t$ holds $r \leq t$. Then $r = \sup X$.

(12) Let X be a non empty real-membered set and Y be a real-membered set. If $X \subseteq Y$ and Y is lower bounded, then $\inf Y \leq \inf X$.

(13) Let X be a non empty real-membered set and Y be a real-membered set. If $X \subseteq Y$ and Y is upper bounded, then $\sup X \leq \sup Y$.

Let X be a real-membered set. We say that X has maximum if and only if:

(Def. 3) X is upper bounded and $\sup X \in X$.

We say that X has minimum if and only if:

(Def. 4) X is lower bounded and $\inf X \in X$.

One can check that there exists a subset of $\mathbb{R}$ which is non empty, closed, and bounded.
Let R be a family of subsets of $\mathbb{R}$. We say that R is open if and only if:

(Def. 5) For every subset X of $\mathbb{R}$ such that $X \in R$ holds X is open.

We say that R is closed if and only if:

(Def. 6) For every subset X of $\mathbb{R}$ such that $X \in R$ holds X is closed.

In the sequel r_3, r_1, r_2, q_3, p_3 denote real numbers.

Let X be a subset of $\mathbb{R}$. The functor $-X$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 7) $-X = \{-r_3 : r_3 \in X\}$.

Let us notice that the functor $-X$ is involutive.

We now state the proposition

(14) $r \in X$ iff $-r \in -X$.

Let X be a non empty subset of $\mathbb{R}$. One can check that $-X$ is non empty.

Next we state several propositions:

(15) X is upper bounded iff $-X$ is lower bounded.

(16) X is lower bounded iff $-X$ is upper bounded.

(17) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded holds $\inf X = -\sup(-X)$.

(18) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded holds $\sup X = -\inf(-X)$.

(19) X is closed iff $-X$ is closed.

Let X be a subset of $\mathbb{R}$ and let p be a real number. The functor $p+X$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 8) $p+X = \{p+r_3 : r_3 \in X\}$.

Next we state the proposition

(20) $r \in X$ iff $q_3+r \in q_3+X$.

Let X be a non empty subset of $\mathbb{R}$ and let s be a real number. One can verify that $s+X$ is non empty.

Next we state several propositions:

(21) $X = 0+X$.

(22) $q_3+(p_3+X) = (q_3+p_3)+X$.

(23) X is upper bounded iff q_3+X is upper bounded.

(24) X is lower bounded iff q_3+X is lower bounded.

(25) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded holds $\inf(q_3+X) = q_3+\inf X$.

(26) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded holds $\sup(q_3+X) = q_3+\sup X$.

(27) X is closed iff q_3+X is closed.

Let X be a subset of $\mathbb{R}$. The functor $\mathrm{Inv}\,X$ yielding a subset of $\mathbb{R}$ is defined as follows:

(Def. 9) $\mathrm{Inv}\,X = \{\frac{1}{r_3} : r_3 \in X\}$.

The following proposition is true

(28) For every without zero subset X of $\mathbb{R}$ holds $r \in X$ iff $\frac{1}{r} \in \mathrm{Inv}\,X$.

Let X be a non empty without zero subset of $\mathbb{R}$. Note that $\mathrm{Inv}\,X$ is non empty and without zero.

Let X be a without zero subset of $\mathbb{R}$. Observe that $\mathrm{Inv}\,X$ is without zero.

The following three propositions are true:

(29) For every without zero subset X of $\mathbb{R}$ holds $\mathrm{Inv}\,\mathrm{Inv}\,X = X$.

(30) For every without zero subset X of $\mathbb{R}$ such that X is closed and bounded holds $\mathrm{Inv}\,X$ is closed.

(31) For every family Z of subsets of $\mathbb{R}$ such that Z is closed holds $\bigcap Z$ is closed.

Let X be a subset of $\mathbb{R}$. The functor $\overline{X}$ yields a subset of $\mathbb{R}$ and is defined by:

(Def. 10) $\overline{X} = \bigcap\{A; A$ ranges over elements of $2^{\mathbb{R}}$: $X \subseteq A \wedge A$is closed$\}$.

Let us note that the functor $\overline{X}$ is projective.

Let X be a subset of $\mathbb{R}$. One can check that $\overline{X}$ is closed.

One can prove the following propositions:

(32) For every closed subset Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\overline{X} \subseteq Y$.

(33) $X \subseteq \overline{X}$.

(34) X is closed iff $X = \overline{X}$.

(35) $\overline{\emptyset_{\mathbb{R}}} = \emptyset$.

(36) $\overline{\Omega_{\mathbb{R}}} = \mathbb{R}$.

(37) If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

(38) $r_3 \in \overline{X}$ iff for every open subset O of $\mathbb{R}$ such that $r_3 \in O$ holds $O \cap X$ is non empty.

(39) If $r_3 \in \overline{X}$, then there exists s_1 such that $\mathrm{rng}\, s_1 \subseteq X$ and s_1 is convergent and $\lim s_1 = r_3$.

2. Functions into Reals

Let X be a set and let f be a function from X into $\mathbb{R}$. Let us observe that f is lower bounded if and only if:

(Def. 11) $f^\circ X$ is lower bounded.

Let us observe that f is upper bounded if and only if:

(Def. 12) $f^\circ X$ is upper bounded.

Let X be a set and let f be a function from X into $\mathbb{R}$. We say that f has maximum if and only if:

(Def. 14)[1] $f^\circ X$ has maximum.

We say that f has minimum if and only if:

(Def. 15) $f^\circ X$ has minimum.

Let X be a set and let f be a function from X into $\mathbb{R}$. The functor $-f$ yielding a function from X into $\mathbb{R}$ is defined as follows:

(Def. 16) For every set p such that $p \in X$ holds $(-f)(p) = -f(p)$.

Let us note that the functor $-f$ is involutive.

One can prove the following propositions:

(40) For all sets X, A and for every function f from X into $\mathbb{R}$ holds $(-f)^\circ A = -f^\circ A$.

(41) For every non empty set X and for every function f from X into $\mathbb{R}$ holds f has minimum iff $-f$ has maximum.

[1] The definition (Def. 13) has been removed.

(42) For every non empty set X and for every function f from X into $\mathbb{R}$ holds f has maximum iff $-f$ has minimum.

(43) For every set X and for every subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ holds $(-f)^{-1}(A) = f^{-1}(-A)$.

Let X be a set, let r be a real number, and let f be a function from X into $\mathbb{R}$. The functor $r+f$ yields a function from X into $\mathbb{R}$ and is defined as follows:

(Def. 17) For every set p such that $p \in X$ holds $(r+f)(p) = r+f(p)$.

One can prove the following propositions:

(44) For all sets X, A and for every function f from X into $\mathbb{R}$ and for every real number s holds $(s+f)^{\circ}A = s+f^{\circ}A$.

(45) For every set X and for every subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ and for every q_3 holds $(q_3+f)^{-1}(A) = f^{-1}(-q_3+A)$.

Let X be a set and let f be a function from X into $\mathbb{R}$. The functor $\mathrm{Inv}\, f$ yields a function from X into $\mathbb{R}$ and is defined as follows:

(Def. 18) For every set p such that $p \in X$ holds $(\mathrm{Inv}\, f)(p) = \frac{1}{f(p)}$.

Let us observe that the functor $\mathrm{Inv}\, f$ is involutive.

The following proposition is true

(46) For every set X and for every without zero subset A of $\mathbb{R}$ and for every function f from X into $\mathbb{R}$ holds $(\mathrm{Inv}\, f)^{-1}(A) = f^{-1}(\mathrm{Inv} A)$.

3. REAL MAPS

Let T be a 1-sorted structure. A real map of T is a function from the carrier of T into $\mathbb{R}$.

Let T be a non empty 1-sorted structure. Observe that there exists a real map of T which is bounded.

In this article we present several logical schemes. The scheme *NonUniqExRF* deals with a non empty topological structure $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a real map f of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the following condition is met:

- For every set x such that $x \in$ the carrier of $\mathcal{A}$ there exists r_3 such that $\mathcal{P}[x, r_3]$.

The scheme *LambdaRF* deals with a non empty topological structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a real number, and states that:

There exists a real map f of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $f(x) = \mathcal{F}(x)$

for all values of the parameters.

Let T be a 1-sorted structure, let f be a real map of T, and let P be a set. Then $f^{-1}(P)$ is a subset of T.

Let T be a 1-sorted structure and let f be a real map of T. The functor $\inf f$ yielding a real number is defined by:

(Def. 20)[2] $\inf f = \inf(f^{\circ}(\text{the carrier of } T))$.

The functor $\sup f$ yields a real number and is defined by:

(Def. 21) $\sup f = \sup(f^{\circ}(\text{the carrier of } T))$.

The following propositions are true:

(47) Let T be a non empty topological space, f be a lower bounded real map of T, and p be a point of T. Then $f(p) \geq \inf f$.

[2] The definition (Def. 19) has been removed.

(48) Let T be a non empty topological space, f be a lower bounded real map of T, and s be a real number. If for every point t of T holds $f(t) \geq s$, then $\inf f \geq s$.

(49) Let T be a non empty topological space and f be a real map of T. Suppose for every point p of T holds $f(p) \geq r$ and for every t such that for every point p of T holds $f(p) \geq t$ holds $r \geq t$. Then $r = \inf f$.

(50) Let T be a non empty topological space, f be an upper bounded real map of T, and p be a point of T. Then $f(p) \leq \sup f$.

(51) Let T be a non empty topological space, f be an upper bounded real map of T, and given t. If for every point p of T holds $f(p) \leq t$, then $\sup f \leq t$.

(52) Let T be a non empty topological space and f be a real map of T. Suppose for every point p of T holds $f(p) \leq r$ and for every t such that for every point p of T holds $f(p) \leq t$ holds $r \leq t$. Then $r = \sup f$.

(53) For every non empty 1-sorted structure T and for every bounded real map f of T holds $\inf f \leq \sup f$.

Let T be a topological structure and let f be a real map of T. We say that f is continuous if and only if:

(Def. 25)[3] For every subset Y of $\mathbb{R}$ such that Y is closed holds $f^{-1}(Y)$ is closed.

Let T be a non empty topological space. Observe that there exists a real map of T which is continuous.

Let T be a non empty topological space and let S be a non empty subspace of T. Observe that there exists a real map of S which is continuous.

In the sequel T denotes a topological structure and f denotes a real map of T.

Next we state four propositions:

(54) f is continuous iff for every subset Y of $\mathbb{R}$ such that Y is open holds $f^{-1}(Y)$ is open.

(55) If f is continuous, then $-f$ is continuous.

(56) If f is continuous, then $r_3 + f$ is continuous.

(57) If f is continuous and $0 \notin \operatorname{rng} f$, then $\operatorname{Inv} f$ is continuous.

Let X, Y be sets, let f be a function from 2^X into 2^Y, and let R be a family of subsets of X. Then $f^\circ R$ is a family of subsets of Y.

The following two propositions are true:

(58) For every family R of subsets of $\mathbb{R}$ such that f is continuous and R is open holds $({}^{-1}f)^\circ R$ is open.

(59) For every family R of subsets of $\mathbb{R}$ such that f is continuous and R is closed holds $({}^{-1}f)^\circ R$ is closed.

Let T be a non empty topological structure, let X be a subset of T, and let f be a real map of T. The functor $f \restriction X$ yielding a real map of $T \restriction X$ is defined by:

(Def. 26) $f \restriction X = f \restriction X$.

Let T be a non empty topological space. Note that there exists a subset of T which is compact and non empty.

Let T be a non empty topological space, let f be a continuous real map of T, and let X be a subset of T. One can verify that $f \restriction X$ is continuous.

Let T be a non empty topological space and let P be a compact non empty subset of T. One can check that $T \restriction P$ is compact.

[3] The definitions (Def. 22)–(Def. 24) have been removed.

4. PSEUDOCOMPACT SPACES

We now state two propositions:

(60) Let T be a non empty topological space. Then for every real map f of T such that f is continuous holds f has maximum if and only if for every real map f of T such that f is continuous holds f has minimum.

(61) Let T be a non empty topological space. Then for every real map f of T such that f is continuous holds f is bounded if and only if for every real map f of T such that f is continuous holds f has maximum.

Let T be a topological structure. We say that T is pseudocompact if and only if:

(Def. 27) For every real map f of T such that f is continuous holds f is bounded.

Let us observe that every non empty topological space which is compact is also pseudocompact. Let us note that there exists a topological space which is compact and non empty.

Let T be a pseudocompact non empty topological space. Note that every real map of T which is continuous is also bounded and has maximum and minimum.

One can prove the following two propositions:

(62) Let T be a non empty topological space, X be a non empty subset of T, Y be a compact subset of T, and f be a continuous real map of T. If $X \subseteq Y$, then $\inf(f \restriction Y) \leq \inf(f \restriction X)$.

(63) Let T be a non empty topological space, X be a non empty subset of T, Y be a compact subset of T, and f be a continuous real map of T. If $X \subseteq Y$, then $\sup(f \restriction X) \leq \sup(f \restriction Y)$.

5. BOUNDING BOXES FOR COMPACT SETS IN $\mathcal{E}^2$

Let n be a natural number and let p_1, p_2 be points of $\mathcal{E}^n_{\mathrm{T}}$. One can check that $\mathcal{L}(p_1, p_2)$ is compact.

We now state the proposition

(64) For every natural number n and for all compact subsets X, Y of $\mathcal{E}^n_{\mathrm{T}}$ holds $X \cap Y$ is compact.

In the sequel p is a point of $\mathcal{E}^2_{\mathrm{T}}$, P is a subset of $\mathcal{E}^2_{\mathrm{T}}$, Z is a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, and X is a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

The real map proj1 of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 28) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{proj1}(p) = p_{\mathbf{1}}$.

The real map proj2 of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 29) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ holds $\mathrm{proj2}(p) = p_{\mathbf{2}}$.

We now state four propositions:

(65) $\mathrm{proj1}^{-1}(]r,s[) = \{[r_1, r_2] : r < r_1 \land r_1 < s\}$.

(66) For all r_3, q_3 such that $P = \{[r_1, r_2] : r_3 < r_1 \land r_1 < q_3\}$ holds P is open.

(67) $\mathrm{proj2}^{-1}(]r,s[) = \{[r_1, r_2] : r < r_2 \land r_2 < s\}$.

(68) For all r_3, q_3 such that $P = \{[r_1, r_2] : r_3 < r_2 \land r_2 < q_3\}$ holds P is open.

One can check that proj1 is continuous and proj2 is continuous.

Next we state two propositions:

(69) For every subset X of $\mathcal{E}^2_{\mathrm{T}}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in X$ holds $(\mathrm{proj1} \restriction X)(p) = p_{\mathbf{1}}$.

(70) For every subset X of $\mathcal{E}^2_{\rm T}$ and for every point p of $\mathcal{E}^2_{\rm T}$ such that $p \in X$ holds $({\rm proj2} \restriction X)(p) = p_{\mathbf{2}}$.

Let X be a subset of $\mathcal{E}^2_{\rm T}$. The functor W-bound(X) yielding a real number is defined as follows:

(Def. 30) W-bound$(X) = \inf({\rm proj1} \restriction X)$.

The functor N-bound(X) yielding a real number is defined as follows:

(Def. 31) N-bound$(X) = \sup({\rm proj2} \restriction X)$.

The functor E-bound(X) yielding a real number is defined by:

(Def. 32) E-bound$(X) = \sup({\rm proj1} \restriction X)$.

The functor S-bound(X) yielding a real number is defined by:

(Def. 33) S-bound$(X) = \inf({\rm proj2} \restriction X)$.

We now state the proposition

(71) If $p \in X$, then W-bound$(X) \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq$ E-bound(X) and S-bound$(X) \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq$ N-bound(X).

Let X be a subset of $\mathcal{E}^2_{\rm T}$. The functor SW-corner(X) yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 34) SW-corner$(X) = [$W-bound(X), S-bound$(X)]$.

The functor NW-corner(X) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 35) NW-corner$(X) = [$W-bound(X), N-bound$(X)]$.

The functor NE-corner(X) yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 36) NE-corner$(X) = [$E-bound(X), N-bound$(X)]$.

The functor SE-corner(X) yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 37) SE-corner$(X) = [$E-bound(X), S-bound$(X)]$.

We now state a number of propositions:

(72) (SW-corner$(P))_{\mathbf{1}} =$ W-bound(P).

(73) (SW-corner$(P))_{\mathbf{2}} =$ S-bound(P).

(74) (NW-corner$(P))_{\mathbf{1}} =$ W-bound(P).

(75) (NW-corner$(P))_{\mathbf{2}} =$ N-bound(P).

(76) (NE-corner$(P))_{\mathbf{1}} =$ E-bound(P).

(77) (NE-corner$(P))_{\mathbf{2}} =$ N-bound(P).

(78) (SE-corner$(P))_{\mathbf{1}} =$ E-bound(P).

(79) (SE-corner$(P))_{\mathbf{2}} =$ S-bound(P).

(80) (SW-corner$(P))_{\mathbf{1}} =$ (NW-corner$(P))_{\mathbf{1}}$.

(81) (SE-corner$(P))_{\mathbf{1}} =$ (NE-corner$(P))_{\mathbf{1}}$.

(82) (NW-corner$(P))_{\mathbf{2}} =$ (NE-corner$(P))_{\mathbf{2}}$.

(83) (SW-corner$(P))_{\mathbf{2}} =$ (SE-corner$(P))_{\mathbf{2}}$.

Let X be a subset of $\mathcal{E}^2_{\rm T}$. The functor ${\rm W}_{\rm most}(X)$ yields a subset of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 38) $\mathrm{W_{most}}(X) = \mathcal{L}(\text{SW-corner}(X), \text{NW-corner}(X)) \cap X$.

The functor $\mathrm{N_{most}}(X)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 39) $\mathrm{N_{most}}(X) = \mathcal{L}(\text{NW-corner}(X), \text{NE-corner}(X)) \cap X$.

The functor $\mathrm{E_{most}}(X)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 40) $\mathrm{E_{most}}(X) = \mathcal{L}(\text{SE-corner}(X), \text{NE-corner}(X)) \cap X$.

The functor $\mathrm{S_{most}}(X)$ yielding a subset of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 41) $\mathrm{S_{most}}(X) = \mathcal{L}(\text{SW-corner}(X), \text{SE-corner}(X)) \cap X$.

Let X be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$. One can verify the following observations:

* $\mathrm{W_{most}}(X)$ is non empty and compact,
* $\mathrm{N_{most}}(X)$ is non empty and compact,
* $\mathrm{E_{most}}(X)$ is non empty and compact, and
* $\mathrm{S_{most}}(X)$ is non empty and compact.

Let X be a subset of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{W_{min}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 42) $\mathrm{W_{min}}(X) = [\text{W-bound}(X), \inf(\text{proj2} \restriction \mathrm{W_{most}}(X))]$.

The functor $\mathrm{W_{max}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 43) $\mathrm{W_{max}}(X) = [\text{W-bound}(X), \sup(\text{proj2} \restriction \mathrm{W_{most}}(X))]$.

The functor $\mathrm{N_{min}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 44) $\mathrm{N_{min}}(X) = [\inf(\text{proj1} \restriction \mathrm{N_{most}}(X)), \text{N-bound}(X)]$.

The functor $\mathrm{N_{max}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 45) $\mathrm{N_{max}}(X) = [\sup(\text{proj1} \restriction \mathrm{N_{most}}(X)), \text{N-bound}(X)]$.

The functor $\mathrm{E_{max}}(X)$ yields a point of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 46) $\mathrm{E_{max}}(X) = [\text{E-bound}(X), \sup(\text{proj2} \restriction \mathrm{E_{most}}(X))]$.

The functor $\mathrm{E_{min}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 47) $\mathrm{E_{min}}(X) = [\text{E-bound}(X), \inf(\text{proj2} \restriction \mathrm{E_{most}}(X))]$.

The functor $\mathrm{S_{max}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 48) $\mathrm{S_{max}}(X) = [\sup(\text{proj1} \restriction \mathrm{S_{most}}(X)), \text{S-bound}(X)]$.

The functor $\mathrm{S_{min}}(X)$ yielding a point of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 49) $\mathrm{S_{min}}(X) = [\inf(\text{proj1} \restriction \mathrm{S_{most}}(X)), \text{S-bound}(X)]$.

One can prove the following propositions:

(84) $(\mathrm{W_{min}}(P))_{\mathbf{1}} = \text{W-bound}(P)$ and $(\mathrm{W_{max}}(P))_{\mathbf{1}} = \text{W-bound}(P)$.

(85) $(\text{SW-corner}(P))_{\mathbf{1}} = (\mathrm{W_{min}}(P))_{\mathbf{1}}$ and $(\text{SW-corner}(P))_{\mathbf{1}} = (\mathrm{W_{max}}(P))_{\mathbf{1}}$ and $(\mathrm{W_{min}}(P))_{\mathbf{1}} = (\mathrm{W_{max}}(P))_{\mathbf{1}}$ and $(\mathrm{W_{min}}(P))_{\mathbf{1}} = (\text{NW-corner}(P))_{\mathbf{1}}$ and $(\mathrm{W_{max}}(P))_{\mathbf{1}} = (\text{NW-corner}(P))_{\mathbf{1}}$.

(86) $(\mathrm{W_{min}}(P))_{\mathbf{2}} = \inf(\text{proj2} \restriction \mathrm{W_{most}}(P))$ and $(\mathrm{W_{max}}(P))_{\mathbf{2}} = \sup(\text{proj2} \restriction \mathrm{W_{most}}(P))$.

(87) $(\text{SW-corner}(X))_{\mathbf{2}} \leq (\mathrm{W_{min}}(X))_{\mathbf{2}}$ and $(\text{SW-corner}(X))_{\mathbf{2}} \leq (\mathrm{W_{max}}(X))_{\mathbf{2}}$ and $(\text{SW-corner}(X))_{\mathbf{2}} \leq (\text{NW-corner}(X))_{\mathbf{2}}$ and $(\mathrm{W_{min}}(X))_{\mathbf{2}} \leq (\mathrm{W_{max}}(X))_{\mathbf{2}}$ and $(\mathrm{W_{min}}(X))_{\mathbf{2}} \leq (\text{NW-corner}(X))_{\mathbf{2}}$ and $(\mathrm{W_{max}}(X))_{\mathbf{2}} \leq (\text{NW-corner}(X))_{\mathbf{2}}$.

(88) If $p \in \mathrm{W}_{\mathrm{most}}(Z)$, then $p_{\mathbf{1}} = (\mathrm{W}_{\mathrm{min}}(Z))_{\mathbf{1}}$ and if Z is compact, then $(\mathrm{W}_{\mathrm{min}}(Z))_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (\mathrm{W}_{\mathrm{max}}(Z))_{\mathbf{2}}$.

(89) $\mathrm{W}_{\mathrm{most}}(X) \subseteq \mathcal{L}(\mathrm{W}_{\mathrm{min}}(X), \mathrm{W}_{\mathrm{max}}(X))$.

(90) $\mathcal{L}(\mathrm{W}_{\mathrm{min}}(X), \mathrm{W}_{\mathrm{max}}(X)) \subseteq \mathcal{L}(\text{SW-corner}(X), \text{NW-corner}(X))$.

(91) $\mathrm{W}_{\mathrm{min}}(X) \in \mathrm{W}_{\mathrm{most}}(X)$ and $\mathrm{W}_{\mathrm{max}}(X) \in \mathrm{W}_{\mathrm{most}}(X)$.

(92) $\mathcal{L}(\text{SW-corner}(X), \mathrm{W}_{\mathrm{min}}(X)) \cap X = \{\mathrm{W}_{\mathrm{min}}(X)\}$ and $\mathcal{L}(\mathrm{W}_{\mathrm{max}}(X), \text{NW-corner}(X)) \cap X = \{\mathrm{W}_{\mathrm{max}}(X)\}$.

(93) If $\mathrm{W}_{\mathrm{min}}(X) = \mathrm{W}_{\mathrm{max}}(X)$, then $\mathrm{W}_{\mathrm{most}}(X) = \{\mathrm{W}_{\mathrm{min}}(X)\}$.

(94) $(\mathrm{N}_{\mathrm{min}}(P))_{\mathbf{2}} = \text{N-bound}(P)$ and $(\mathrm{N}_{\mathrm{max}}(P))_{\mathbf{2}} = \text{N-bound}(P)$.

(95) $(\text{NW-corner}(P))_{\mathbf{2}} = (\mathrm{N}_{\mathrm{min}}(P))_{\mathbf{2}}$ and $(\text{NW-corner}(P))_{\mathbf{2}} = (\mathrm{N}_{\mathrm{max}}(P))_{\mathbf{2}}$ and $(\mathrm{N}_{\mathrm{min}}(P))_{\mathbf{2}} = (\mathrm{N}_{\mathrm{max}}(P))_{\mathbf{2}}$ and $(\mathrm{N}_{\mathrm{min}}(P))_{\mathbf{2}} = (\text{NE-corner}(P))_{\mathbf{2}}$ and $(\mathrm{N}_{\mathrm{max}}(P))_{\mathbf{2}} = (\text{NE-corner}(P))_{\mathbf{2}}$.

(96) $(\mathrm{N}_{\mathrm{min}}(P))_{\mathbf{1}} = \inf(\mathrm{proj1} \restriction \mathrm{N}_{\mathrm{most}}(P))$ and $(\mathrm{N}_{\mathrm{max}}(P))_{\mathbf{1}} = \sup(\mathrm{proj1} \restriction \mathrm{N}_{\mathrm{most}}(P))$.

(97) $(\text{NW-corner}(X))_{\mathbf{1}} \leq (\mathrm{N}_{\mathrm{min}}(X))_{\mathbf{1}}$ and $(\text{NW-corner}(X))_{\mathbf{1}} \leq (\mathrm{N}_{\mathrm{max}}(X))_{\mathbf{1}}$ and $(\text{NW-corner}(X))_{\mathbf{1}} \leq (\text{NE-corner}(X))_{\mathbf{1}}$ and $(\mathrm{N}_{\mathrm{min}}(X))_{\mathbf{1}} \leq (\mathrm{N}_{\mathrm{max}}(X))_{\mathbf{1}}$ and $(\mathrm{N}_{\mathrm{min}}(X))_{\mathbf{1}} \leq (\text{NE-corner}(X))_{\mathbf{1}}$ and $(\mathrm{N}_{\mathrm{max}}(X))_{\mathbf{1}} \leq (\text{NE-corner}(X))_{\mathbf{1}}$.

(98) If $p \in \mathrm{N}_{\mathrm{most}}(Z)$, then $p_{\mathbf{2}} = (\mathrm{N}_{\mathrm{min}}(Z))_{\mathbf{2}}$ and if Z is compact, then $(\mathrm{N}_{\mathrm{min}}(Z))_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (\mathrm{N}_{\mathrm{max}}(Z))_{\mathbf{1}}$.

(99) $\mathrm{N}_{\mathrm{most}}(X) \subseteq \mathcal{L}(\mathrm{N}_{\mathrm{min}}(X), \mathrm{N}_{\mathrm{max}}(X))$.

(100) $\mathcal{L}(\mathrm{N}_{\mathrm{min}}(X), \mathrm{N}_{\mathrm{max}}(X)) \subseteq \mathcal{L}(\text{NW-corner}(X), \text{NE-corner}(X))$.

(101) $\mathrm{N}_{\mathrm{min}}(X) \in \mathrm{N}_{\mathrm{most}}(X)$ and $\mathrm{N}_{\mathrm{max}}(X) \in \mathrm{N}_{\mathrm{most}}(X)$.

(102) $\mathcal{L}(\text{NW-corner}(X), \mathrm{N}_{\mathrm{min}}(X)) \cap X = \{\mathrm{N}_{\mathrm{min}}(X)\}$ and $\mathcal{L}(\mathrm{N}_{\mathrm{max}}(X), \text{NE-corner}(X)) \cap X = \{\mathrm{N}_{\mathrm{max}}(X)\}$.

(103) If $\mathrm{N}_{\mathrm{min}}(X) = \mathrm{N}_{\mathrm{max}}(X)$, then $\mathrm{N}_{\mathrm{most}}(X) = \{\mathrm{N}_{\mathrm{min}}(X)\}$.

(104) $(\mathrm{E}_{\mathrm{min}}(P))_{\mathbf{1}} = \text{E-bound}(P)$ and $(\mathrm{E}_{\mathrm{max}}(P))_{\mathbf{1}} = \text{E-bound}(P)$.

(105) $(\text{SE-corner}(P))_{\mathbf{1}} = (\mathrm{E}_{\mathrm{min}}(P))_{\mathbf{1}}$ and $(\text{SE-corner}(P))_{\mathbf{1}} = (\mathrm{E}_{\mathrm{max}}(P))_{\mathbf{1}}$ and $(\mathrm{E}_{\mathrm{min}}(P))_{\mathbf{1}} = (\mathrm{E}_{\mathrm{max}}(P))_{\mathbf{1}}$ and $(\mathrm{E}_{\mathrm{min}}(P))_{\mathbf{1}} = (\text{NE-corner}(P))_{\mathbf{1}}$ and $(\mathrm{E}_{\mathrm{max}}(P))_{\mathbf{1}} = (\text{NE-corner}(P))_{\mathbf{1}}$.

(106) $(\mathrm{E}_{\mathrm{min}}(P))_{\mathbf{2}} = \inf(\mathrm{proj2} \restriction \mathrm{E}_{\mathrm{most}}(P))$ and $(\mathrm{E}_{\mathrm{max}}(P))_{\mathbf{2}} = \sup(\mathrm{proj2} \restriction \mathrm{E}_{\mathrm{most}}(P))$.

(107) $(\text{SE-corner}(X))_{\mathbf{2}} \leq (\mathrm{E}_{\mathrm{min}}(X))_{\mathbf{2}}$ and $(\text{SE-corner}(X))_{\mathbf{2}} \leq (\mathrm{E}_{\mathrm{max}}(X))_{\mathbf{2}}$ and $(\text{SE-corner}(X))_{\mathbf{2}} \leq (\text{NE-corner}(X))_{\mathbf{2}}$ and $(\mathrm{E}_{\mathrm{min}}(X))_{\mathbf{2}} \leq (\mathrm{E}_{\mathrm{max}}(X))_{\mathbf{2}}$ and $(\mathrm{E}_{\mathrm{min}}(X))_{\mathbf{2}} \leq (\text{NE-corner}(X))_{\mathbf{2}}$ and $(\mathrm{E}_{\mathrm{max}}(X))_{\mathbf{2}} \leq (\text{NE-corner}(X))_{\mathbf{2}}$.

(108) If $p \in \mathrm{E}_{\mathrm{most}}(Z)$, then $p_{\mathbf{1}} = (\mathrm{E}_{\mathrm{min}}(Z))_{\mathbf{1}}$ and if Z is compact, then $(\mathrm{E}_{\mathrm{min}}(Z))_{\mathbf{2}} \leq p_{\mathbf{2}}$ and $p_{\mathbf{2}} \leq (\mathrm{E}_{\mathrm{max}}(Z))_{\mathbf{2}}$.

(109) $\mathrm{E}_{\mathrm{most}}(X) \subseteq \mathcal{L}(\mathrm{E}_{\mathrm{min}}(X), \mathrm{E}_{\mathrm{max}}(X))$.

(110) $\mathcal{L}(\mathrm{E}_{\mathrm{min}}(X), \mathrm{E}_{\mathrm{max}}(X)) \subseteq \mathcal{L}(\text{SE-corner}(X), \text{NE-corner}(X))$.

(111) $\mathrm{E}_{\mathrm{min}}(X) \in \mathrm{E}_{\mathrm{most}}(X)$ and $\mathrm{E}_{\mathrm{max}}(X) \in \mathrm{E}_{\mathrm{most}}(X)$.

(112) $\mathcal{L}(\text{SE-corner}(X), \mathrm{E}_{\mathrm{min}}(X)) \cap X = \{\mathrm{E}_{\mathrm{min}}(X)\}$ and $\mathcal{L}(\mathrm{E}_{\mathrm{max}}(X), \text{NE-corner}(X)) \cap X = \{\mathrm{E}_{\mathrm{max}}(X)\}$.

(113) If $\mathrm{E}_{\mathrm{min}}(X) = \mathrm{E}_{\mathrm{max}}(X)$, then $\mathrm{E}_{\mathrm{most}}(X) = \{\mathrm{E}_{\mathrm{min}}(X)\}$.

(114) $(\mathrm{S}_{\mathrm{min}}(P))_{\mathbf{2}} = \text{S-bound}(P)$ and $(\mathrm{S}_{\mathrm{max}}(P))_{\mathbf{2}} = \text{S-bound}(P)$.

(115) $(\text{SW-corner}(P))_{\mathbf{2}} = (\text{S}_{\text{min}}(P))_{\mathbf{2}}$ and $(\text{SW-corner}(P))_{\mathbf{2}} = (\text{S}_{\text{max}}(P))_{\mathbf{2}}$ and $(\text{S}_{\text{min}}(P))_{\mathbf{2}} = (\text{S}_{\text{max}}(P))_{\mathbf{2}}$ and $(\text{S}_{\text{min}}(P))_{\mathbf{2}} = (\text{SE-corner}(P))_{\mathbf{2}}$ and $(\text{S}_{\text{max}}(P))_{\mathbf{2}} = (\text{SE-corner}(P))_{\mathbf{2}}$.

(116) $(\text{S}_{\text{min}}(P))_{\mathbf{1}} = \inf(\text{proj1} \restriction \text{S}_{\text{most}}(P))$ and $(\text{S}_{\text{max}}(P))_{\mathbf{1}} = \sup(\text{proj1} \restriction \text{S}_{\text{most}}(P))$.

(117) $(\text{SW-corner}(X))_{\mathbf{1}} \leq (\text{S}_{\text{min}}(X))_{\mathbf{1}}$ and $(\text{SW-corner}(X))_{\mathbf{1}} \leq (\text{S}_{\text{max}}(X))_{\mathbf{1}}$ and $(\text{SW-corner}(X))_{\mathbf{1}} \leq (\text{SE-corner}(X))_{\mathbf{1}}$ and $(\text{S}_{\text{min}}(X))_{\mathbf{1}} \leq (\text{S}_{\text{max}}(X))_{\mathbf{1}}$ and $(\text{S}_{\text{min}}(X))_{\mathbf{1}} \leq (\text{SE-corner}(X))_{\mathbf{1}}$ and $(\text{S}_{\text{max}}(X))_{\mathbf{1}} \leq (\text{SE-corner}(X))_{\mathbf{1}}$.

(118) If $p \in \text{S}_{\text{most}}(Z)$, then $p_{\mathbf{2}} = (\text{S}_{\text{min}}(Z))_{\mathbf{2}}$ and if Z is compact, then $(\text{S}_{\text{min}}(Z))_{\mathbf{1}} \leq p_{\mathbf{1}}$ and $p_{\mathbf{1}} \leq (\text{S}_{\text{max}}(Z))_{\mathbf{1}}$.

(119) $\text{S}_{\text{most}}(X) \subseteq \mathcal{L}(\text{S}_{\text{min}}(X), \text{S}_{\text{max}}(X))$.

(120) $\mathcal{L}(\text{S}_{\text{min}}(X), \text{S}_{\text{max}}(X)) \subseteq \mathcal{L}(\text{SW-corner}(X), \text{SE-corner}(X))$.

(121) $\text{S}_{\text{min}}(X) \in \text{S}_{\text{most}}(X)$ and $\text{S}_{\text{max}}(X) \in \text{S}_{\text{most}}(X)$.

(122) $\mathcal{L}(\text{SW-corner}(X), \text{S}_{\text{min}}(X)) \cap X = \{\text{S}_{\text{min}}(X)\}$ and $\mathcal{L}(\text{S}_{\text{max}}(X), \text{SE-corner}(X)) \cap X = \{\text{S}_{\text{max}}(X)\}$.

(123) If $\text{S}_{\text{min}}(X) = \text{S}_{\text{max}}(X)$, then $\text{S}_{\text{most}}(X) = \{\text{S}_{\text{min}}(X)\}$.

(124) If $\text{W}_{\text{max}}(P) = \text{N}_{\text{min}}(P)$, then $\text{W}_{\text{max}}(P) = \text{NW-corner}(P)$.

(125) If $\text{N}_{\text{max}}(P) = \text{E}_{\text{max}}(P)$, then $\text{N}_{\text{max}}(P) = \text{NE-corner}(P)$.

(126) If $\text{E}_{\text{min}}(P) = \text{S}_{\text{max}}(P)$, then $\text{E}_{\text{min}}(P) = \text{SE-corner}(P)$.

(127) If $\text{S}_{\text{min}}(P) = \text{W}_{\text{min}}(P)$, then $\text{S}_{\text{min}}(P) = \text{SW-corner}(P)$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[2] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/pcomps_1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/compts_1.html.

[6] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[7] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[8] Agata Darmochwał. The Euclidean space. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/euclid.html.

[9] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/topreal1.html.

[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/real_1.html.

[11] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_4.html.

[12] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_2.html.

[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seq_1.html.

[14] M.G. Murdeshwar. *General Topology*. Wiley Eastern, 1990.

[15] Yatsuka Nakamura and Andrzej Trybulec. A mathematical model of CPU. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/ami_1.html.

[16] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[17] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[18] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/absvalue.html.

[19] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rcomp_1.html.

[20] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[21] Andrzej Trybulec. On the sets inhabited by numbers. *Journal of Formalized Mathematics*, 15, 2003. http://mizar.org/JFM/Vol16/membered.html.

[22] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[23] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[24] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received July 29, 1997

Published January 2, 2004