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Product of Families of Groups and Vector Spaces

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Summary. In the first section we present properties of fields and Abelian groups in terms of commutativity, associativity, etc. Next, we are concerned with operations on *n*-tuples on some set which are generalization of operations on this set. It is used in third section to introduce the *n*-power of a group and the *n*-power of a field. Besides, we introduce a concept of indexed family of binary (unary) operations over some indexed family of sets and a product of such families which is binary (unary) operation on a product of family sets. We use that product in the last section to introduce the product of a finite sequence of Abelian groups.

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The articles [13], [7], [16], [1], [17], [5], [6], [4], [11], [15], [10], [8], [3], [14], [2], [12], and [9] provide the notation and terminology for this paper.

1. Abelian Groups and Fields

In this paper G is an Abelian add-associative right complementable right zeroed non empty loop structure.

One can prove the following two propositions:

(3)[1] The zero of G is a unity w.r.t. the addition of G.

(4) For every Abelian group G holds comp G is an inverse operation w.r.t. the addition of G.

In the sequel G_1 is a non empty loop structure.

One can prove the following proposition

(5) Suppose that
 (i) the addition of G_1 is commutative and associative,
 (ii) the zero of G_1 is a unity w.r.t. the addition of G_1, and
 (iii) comp G_1 is an inverse operation w.r.t. the addition of G_1.

 Then G_1 is an Abelian group.

In the sequel F denotes a field.

Next we state two propositions:

(10)[2] The zero of F is a unity w.r.t. the addition of F.

(11) The unity of F is a unity w.r.t. the multiplication of F.

[1] The propositions (1) and (2) have been removed.
[2] The propositions (6)–(9) have been removed.

2. The n-Product of a Binary and a Unary Operation

For simplicity, we follow the rules: F denotes a field, n denotes a natural number, D denotes a non empty set, d denotes an element of D, B denotes a binary operation on D, and C denotes a unary operation on D.

Let us consider D, n, let F be a binary operation on D, and let x, y be elements of D^n. Then $F^\circ(x,y)$ is an element of D^n.

Let D be a non empty set, let F be a binary operation on D, and let n be a natural number. The functor $\pi^n F$ yields a binary operation on D^n and is defined by:

(Def. 1) For all elements x, y of D^n holds $(\pi^n F)(x,y) = F^\circ(x,y)$.

Let us consider D, let F be a unary operation on D, and let us consider n. The functor $\pi^n F$ yielding a unary operation on D^n is defined by:

(Def. 2) For every element x of D^n holds $(\pi^n F)(x) = F \cdot x$.

Let D be a non empty set, let us consider n, and let x be an element of D. Then $n \mapsto x$ is an element of D^n. We introduce $n \dot{\longmapsto} x$ as a synonym of $n \mapsto x$.

One can prove the following four propositions:

(14)[3] If B is commutative, then $\pi^n B$ is commutative.

(15) If B is associative, then $\pi^n B$ is associative.

(16) If d is a unity w.r.t. B, then $n \dot{\longmapsto} d$ is a unity w.r.t. $\pi^n B$.

(17) If B is associative and has a unity and C is an inverse operation w.r.t. B, then $\pi^n C$ is an inverse operation w.r.t. $\pi^n B$.

3. The n-Power of a Group and of a Field

Let F be a non empty loop structure and let us consider n. Let us assume that F is Abelian, add-associative, right zeroed, and right complementable. The functor F^n yields a strict Abelian group and is defined as follows:

(Def. 3) $F^n = \langle$(the carrier of $F)^n$, π^n(the addition of F),($n \dot{\longmapsto}$the zero of F **qua** element of (the carrier of $F)^n$)$\rangle$.

Let F be an Abelian group and let us consider n. Observe that F^n is non empty.

Let us consider F, n. The functor $\cdot^n_F$ yielding a function from [: the carrier of F, (the carrier of $F)^n$:] into (the carrier of $F)^n$ is defined by the condition (Def. 4).

(Def. 4) Let x be an element of F and v be an element of (the carrier of $F)^n$. Then $(\cdot^n_F)(x,v) =$ (the multiplication of $F)^\circ(x,v)$.

Let us consider F, n. The n-dimension vector space over F yields a strict vector space structure over F and is defined by the conditions (Def. 5).

(Def. 5)(i) The loop structure of the n-dimension vector space over $F = F^n$, and

(ii) the left multiplication of the n-dimension vector space over $F = \cdot^n_F$.

Let us consider F, n. Observe that the n-dimension vector space over F is non empty.

For simplicity, we use the following convention: D denotes a non empty set, H, G denote binary operations on D, d denotes an element of D, and t_1, t_2 denote elements of D^n.

One can prove the following proposition

(18) If H is distributive w.r.t. G, then $H^\circ(d, G^\circ(t_1,t_2)) = G^\circ(H^\circ(d,t_1), H^\circ(d,t_2))$.

Let D be a non empty set, let n be a natural number, let F be a binary operation on D, let x be an element of D, and let v be an element of D^n. Then $F^\circ(x,v)$ is an element of D^n.

Let us consider F, n. Note that the n-dimension vector space over F is vector space-like.

[3] The propositions (12) and (13) have been removed.

4. Sequences of Non-empty Sets

In the sequel x is a set.

Let us observe that there exists a finite sequence which is non empty and non-empty.

Let f be a non-empty function. One can verify that $\prod f$ is functional and non empty.

A sequence of non empty sets is a non empty non-empty finite sequence.

Let a be a non empty function. One can verify that $\operatorname{dom} a$ is non empty.

The scheme *NEFinSeqLambda* deals with a non empty finite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a set, and states that:

> There exists a non empty finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and for every element i of $\operatorname{dom} \mathcal{A}$ holds $p(i) = \mathcal{F}(i)$

for all values of the parameters.

Let a be a non-empty non empty function and let i be an element of $\operatorname{dom} a$. One can verify that $a(i)$ is non empty.

Let a be a non-empty non empty function, let f be an element of $\prod a$, and let i be an element of $\operatorname{dom} a$. Then $f(i)$ is an element of $a(i)$.

5. The Product of Families of Operations

In the sequel a is a sequence of non empty sets, i is an element of $\operatorname{dom} a$, and p is a finite sequence.

Let a be a non-empty non empty function. A function is called a family of binary operations of a if:

(Def. 8)[4] $\operatorname{dom} \mathrm{it} = \operatorname{dom} a$ and for every element i of $\operatorname{dom} a$ holds $\mathrm{it}(i)$ is a binary operation on $a(i)$.

A function is called a family of unary operations of a if:

(Def. 9) $\operatorname{dom} \mathrm{it} = \operatorname{dom} a$ and for every element i of $\operatorname{dom} a$ holds $\mathrm{it}(i)$ is a unary operation on $a(i)$.

Let us consider a. Note that every family of binary operations of a is finite sequence-like and every family of unary operations of a is finite sequence-like.

Next we state two propositions:

(19) p is a family of binary operations of a if and only if $\operatorname{len} p = \operatorname{len} a$ and for every i holds $p(i)$ is a binary operation on $a(i)$.

(20) p is a family of unary operations of a if and only if $\operatorname{len} p = \operatorname{len} a$ and for every i holds $p(i)$ is a unary operation on $a(i)$.

Let us consider a, let b be a family of binary operations of a, and let us consider i. Then $b(i)$ is a binary operation on $a(i)$.

Let us consider a, let u be a family of unary operations of a, and let us consider i. Then $u(i)$ is a unary operation on $a(i)$.

Let F be a functional non empty set, let u be a unary operation on F, and let f be an element of F. Then $u(f)$ is an element of F.

One can prove the following two propositions:

(21) Let d, d' be unary operations on $\prod a$. Suppose that for every element f of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $d(f)(i) = d'(f)(i)$. Then $d = d'$.

(22) For every family u of unary operations of a holds $\operatorname{dom}_\kappa u(\kappa) = a$ and $\prod(\operatorname{rng}_\kappa u(\kappa)) \subseteq \prod a$.

Let us consider a and let u be a family of unary operations of a. Then $\mathrm{Frege}(u)$ is a unary operation on $\prod a$.

We now state the proposition

(23) Let u be a family of unary operations of a, f be an element of $\prod a$, and i be an element of $\operatorname{dom} a$. Then $(\mathrm{Frege}(u))(f)(i) = u(i)(f(i))$.

[4] The definitions (Def. 6) and (Def. 7) have been removed.

Let F be a functional non empty set, let b be a binary operation on F, and let f, g be elements of F. Then $b(f,g)$ is an element of F.

We now state the proposition

(24) Let d, d' be binary operations on $\prod a$. Suppose that for all elements f, g of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $d(f,g)(i) = d'(f,g)(i)$. Then $d = d'$.

In the sequel i denotes an element of $\operatorname{dom} a$.

Let us consider a and let b be a family of binary operations of a. The functor $\prod^{\circ} b$ yields a binary operation on $\prod a$ and is defined as follows:

(Def. 10) For all elements f, g of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $(\prod^{\circ} b)(f,g)(i) = b(i)(f(i), g(i))$.

Next we state four propositions:

(25) For every family b of binary operations of a such that for every i holds $b(i)$ is commutative holds $\prod^{\circ} b$ is commutative.

(26) For every family b of binary operations of a such that for every i holds $b(i)$ is associative holds $\prod^{\circ} b$ is associative.

(27) Let b be a family of binary operations of a and f be an element of $\prod a$. If for every i holds $f(i)$ is a unity w.r.t. $b(i)$, then f is a unity w.r.t. $\prod^{\circ} b$.

(28) Let b be a family of binary operations of a and u be a family of unary operations of a. Suppose that for every i holds $u(i)$ is an inverse operation w.r.t. $b(i)$ and $b(i)$ has a unity. Then $\operatorname{Frege}(u)$ is an inverse operation w.r.t. $\prod^{\circ} b$.

6. The Product of Families of Groups

Let F be a function. We say that F is Abelian group yielding if and only if:

(Def. 11) If $x \in \operatorname{rng} F$, then x is an Abelian group.

Let us note that there exists a finite sequence which is non empty and Abelian group yielding.

A sequence of groups is a non empty Abelian group yielding finite sequence.

Let g be a sequence of groups and let i be an element of $\operatorname{dom} g$. Then $g(i)$ is an Abelian group.

Let g be a sequence of groups. The functor $\overline{g}$ yields a sequence of non empty sets and is defined by:

(Def. 12) $\operatorname{len} \overline{g} = \operatorname{len} g$ and for every element j of $\operatorname{dom} g$ holds $\overline{g}(j) =$ the carrier of $g(j)$.

In the sequel g is a sequence of groups and i is an element of $\operatorname{dom} \overline{g}$.

Let us consider g, i. Then $g(i)$ is an Abelian group.

Let us consider g. The functor $\langle +_{g_i} \rangle_i$ yielding a family of binary operations of $\overline{g}$ is defined by:

(Def. 13) $\operatorname{len}(\langle +_{g_i} \rangle_i) = \operatorname{len} \overline{g}$ and for every i holds $\langle +_{g_i} \rangle_i(i) =$ the addition of $g(i)$.

The functor $\langle -_{g_i} \rangle_i$ yielding a family of unary operations of $\overline{g}$ is defined as follows:

(Def. 14) $\operatorname{len}(\langle -_{g_i} \rangle_i) = \operatorname{len} \overline{g}$ and for every i holds $\langle -_{g_i} \rangle_i(i) = \operatorname{comp} g(i)$.

The functor $\langle 0_{g_i} \rangle_i$ yields an element of $\prod \overline{g}$ and is defined as follows:

(Def. 15) For every i holds $\langle 0_{g_i} \rangle_i(i) =$ the zero of $g(i)$.

Let G be a sequence of groups. The functor $\prod G$ yielding a strict Abelian group is defined as follows:

(Def. 16) $\prod G = \langle \prod \overline{G}, \prod^{\circ}(\langle +_{G_i} \rangle_i), \langle 0_{G_i} \rangle_i \rangle$.

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