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Probability

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Summary. Some further theorems concerning probability, among them the equivalent definition of probability are discussed, followed by notions of independence of events and conditional probability and basic theorems on them.

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The articles [10], [5], [12], [2], [11], [13], [6], [3], [4], [8], [7], [9], and [1] provide the notation and terminology for this paper.

For simplicity, we use the following convention: O_1 is a non empty set, m, n are natural numbers, x, y are sets, r, r_1, r_2, r_3 are real numbers, s_1, s_2 are sequences of real numbers, S_1 is a σ-field of subsets of O_1, A_1, B_1 are sequences of subsets of S_1, P, P_1 are probabilities on S_1, and A, B, C, A_2, A_3, A_4 are events of S_1.

One can prove the following three propositions:

(4)[1] For all r, r_1, r_2, r_3 such that $r \neq 0$ and $r_1 \neq 0$ holds $\frac{r_3}{r_1} = \frac{r_2}{r}$ iff $r_3 \cdot r = r_2 \cdot r_1$.

(5) If s_1 is convergent and for every n holds $s_2(n) = r - s_1(n)$, then s_2 is convergent and $\lim s_2 = r - \lim s_1$.

(6) $A \cap O_1 = A$ and $A \cap \Omega_{(S_1)} = A$.

The scheme *SeqExProb* deals with a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function f such that $\mathrm{dom}\, f = \mathbb{N}$ and for every n holds $f(n) = \mathcal{F}(n)$

for all values of the parameter.

Let us consider O_1, S_1, A_1, n. Then $A_1(n)$ is an event of S_1.

Let us consider O_1, S_1, A_1. The functor $\bigcap A_1$ yields an event of S_1 and is defined as follows:

(Def. 1) $\bigcap A_1 = \mathrm{Intersection} A_1$.

We now state several propositions:

(9)[2] There exists B_1 such that for every n holds $B_1(n) = A_1(n) \cap B$.

(10) If A_1 is non-increasing and for every n holds $B_1(n) = A_1(n) \cap B$, then B_1 is non-increasing.

(11) For every function f from S_1 into $\mathbb{R}$ holds $(f \cdot A_1)(n) = f(A_1(n))$.

(12) If for every n holds $B_1(n) = A_1(n) \cap B$, then $\mathrm{Intersection} A_1 \cap B = \mathrm{Intersection} B_1$.

[1] The propositions (1)–(3) have been removed.
[2] The propositions (7) and (8) have been removed.

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(13) If for every A holds $P(A) = P_1(A)$, then $P = P_1$.

(14) For every sequence A_1 of subsets of O_1 holds A_1 is non-increasing iff for every n holds $A_1(n+1) \subseteq A_1(n)$.

(15) For every sequence A_1 of subsets of O_1 holds A_1 is non-decreasing iff for every n holds $A_1(n) \subseteq A_1(n+1)$.

(16) For all sequences A_1, B_1 of subsets of O_1 such that for every n holds $A_1(n) = B_1(n)$ holds $A_1 = B_1$.

(17) For every sequence A_1 of subsets of O_1 holds A_1 is non-increasing iff ComplementA_1 is non-decreasing.

Let us consider O_1, S_1, A_1. The functor ${A_1}^{\mathbf{c}}$ yielding a sequence of subsets of S_1 is defined by:

(Def. 2) ${A_1}^{\mathbf{c}} =$ ComplementA_1.

Let F be a function. We say that F is disjoint valued if and only if:

(Def. 3) If $x \neq y$, then $F(x)$ misses $F(y)$.

Let us consider O_1, S_1, A_1. Let us observe that A_1 is disjoint valued if and only if:

(Def. 4) For all m, n such that $m \neq n$ holds $A_1(m)$ misses $A_1(n)$.

One can prove the following propositions:

(20)[3] Let P be a function from S_1 into $\mathbb{R}$. Then P is a probability on S_1 if and only if the following conditions are satisfied:

(i) for every A holds $0 \leq P(A)$,

(ii) $P(O_1) = 1$,

(iii) for all A, B such that A misses B holds $P(A \cup B) = P(A) + P(B)$, and

(iv) for every A_1 such that A_1 is non-decreasing holds $P \cdot A_1$ is convergent and $\lim(P \cdot A_1) = P(\bigcup A_1)$.

(21) $P(A \cup B \cup C) = ((P(A) + P(B) + P(C)) - (P(A \cap B) + P(B \cap C) + P(A \cap C))) + P(A \cap B \cap C)$.

(22) $P(A \setminus A \cap B) = P(A) - P(A \cap B)$.

(23) $P(A \cap B) \leq P(B)$ and $P(A \cap B) \leq P(A)$.

(24) If $C = B^{\mathrm{c}}$, then $P(A) = P(A \cap B) + P(A \cap C)$.

(25) $(P(A) + P(B)) - 1 \leq P(A \cap B)$.

(26) $P(A) = 1 - P(\Omega_{(S_1)} \setminus A)$.

(27) $P(A) < 1$ iff $0 < P(\Omega_{(S_1)} \setminus A)$.

(28) $P(\Omega_{(S_1)} \setminus A) < 1$ iff $0 < P(A)$.

Let us consider O_1, S_1, P, A, B. We say that A and B are independent w.r.t P if and only if:

(Def. 5) $P(A \cap B) = P(A) \cdot P(B)$.

Let us consider C. We say that A, B and C are independent w.r.t P if and only if:

(Def. 6) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ and $P(A \cap B) = P(A) \cdot P(B)$ and $P(A \cap C) = P(A) \cdot P(C)$ and $P(B \cap C) = P(B) \cdot P(C)$.

[3] The propositions (18) and (19) have been removed.

We now state a number of propositions:

(31)[4] A and B are independent w.r.t P iff B and A are independent w.r.t P.

(32) A, B and C are independent w.r.t P if and only if the following conditions are satisfied:

(i) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$,

(ii) A and B are independent w.r.t P,

(iii) B and C are independent w.r.t P, and

(iv) A and C are independent w.r.t P.

(33) If A, B and C are independent w.r.t P, then B, A and C are independent w.r.t P.

(34) If A, B and C are independent w.r.t P, then A, C and B are independent w.r.t P.

(35) For every event E of S_1 such that $E = \emptyset$ holds A and E are independent w.r.t P.

(36) A and $\Omega_{(S_1)}$ are independent w.r.t P.

(37) For all A, B, P such that A and B are independent w.r.t P holds A and $\Omega_{(S_1)} \setminus B$ are independent w.r.t P.

(38) If A and B are independent w.r.t P, then $\Omega_{(S_1)} \setminus A$ and $\Omega_{(S_1)} \setminus B$ are independent w.r.t P.

(39) Let given A, B, C, P. Suppose A and B are independent w.r.t P and A and C are independent w.r.t P and B misses C. Then A and $B \cup C$ are independent w.r.t P.

(40) For all P, A, B such that A and B are independent w.r.t P and $P(A) < 1$ and $P(B) < 1$ holds $P(A \cup B) < 1$.

Let us consider O_1, S_1, P, B. Let us assume that $0 < P(B)$. The functor $(P|B)$ yields a probability on S_1 and is defined as follows:

(Def. 7) For every A holds $(P|B)(A) = \frac{P(A \cap B)}{P(B)}$.

The following propositions are true:

(42)[5] For all P, B, A such that $0 < P(B)$ holds $P(A \cap B) = (P|B)(A) \cdot P(B)$.

(43) For all P, A, B, C such that $0 < P(A \cap B)$ holds $P(A \cap B \cap C) = P(A) \cdot (P|A)(B) \cdot (P|(A \cap B))(C)$.

(44) For all P, A, B, C such that $C = B^{\rm c}$ and $0 < P(B)$ and $0 < P(C)$ holds $P(A) = (P|B)(A) \cdot P(B) + (P|C)(A) \cdot P(C)$.

(45) Let given P, A, A_2, A_3, A_4. Suppose A_2 misses A_3 and $A_4 = (A_2 \cup A_3)^{\rm c}$ and $0 < P(A_2)$ and $0 < P(A_3)$ and $0 < P(A_4)$. Then $P(A) = (P|A_2)(A) \cdot P(A_2) + (P|A_3)(A) \cdot P(A_3) + (P|A_4)(A) \cdot P(A_4)$.

(46) For all P, A, B such that $0 < P(B)$ holds $(P|B)(A) = P(A)$ iff A and B are independent w.r.t P.

(47) For all P, A, B such that $0 < P(B)$ and $P(B) < 1$ and $(P|B)(A) = (P|(\Omega_{(S_1)} \setminus B))(A)$ holds A and B are independent w.r.t P.

(48) For all P, A, B such that $0 < P(B)$ holds $\frac{(P(A)+P(B))-1}{P(B)} \leq (P|B)(A)$.

(49) For all A, B, P such that $0 < P(A)$ and $0 < P(B)$ holds $(P|B)(A) = \frac{(P|A)(B) \cdot P(A)}{P(B)}$.

[4] The propositions (29) and (30) have been removed.
[5] The proposition (41) has been removed.

(50) Let given B, A_2, A_3, P. Suppose $0 < P(B)$ and $A_3 = {A_2}^{\rm c}$ and $0 < P(A_2)$ and $0 < P(A_3)$. Then $(P|B)(A_2) = \frac{(P|A_2)(B) \cdot P(A_2)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3)}$ and $(P|B)(A_3) = \frac{(P|A_3)(B) \cdot P(A_3)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3)}$.

(51) Let given B, A_2, A_3, A_4, P. Suppose $0 < P(B)$ and $0 < P(A_2)$ and $0 < P(A_3)$ and $0 < P(A_4)$ and A_2 misses A_3 and $A_4 = (A_2 \cup A_3)^{\rm c}$. Then

(i) $(P|B)(A_2) = \frac{(P|A_2)(B) \cdot P(A_2)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3) + (P|A_4)(B) \cdot P(A_4)}$,

(ii) $(P|B)(A_3) = \frac{(P|A_3)(B) \cdot P(A_3)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3) + (P|A_4)(B) \cdot P(A_4)}$, and

(iii) $(P|B)(A_4) = \frac{(P|A_4)(B) \cdot P(A_4)}{(P|A_2)(B) \cdot P(A_2) + (P|A_3)(B) \cdot P(A_3) + (P|A_4)(B) \cdot P(A_4)}$.

(52) For all A, B, P such that $0 < P(B)$ holds $1 - \frac{P(\Omega_{(S_1)} \setminus A)}{P(B)} \leq (P|B)(A)$.

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