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On Paracompactness of Metrizable Spaces

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Summary. The aim is to prove, using Mizar System, one of the most important result in general topology, namely the Stone Theorem on paracompactness of metrizable spaces [18]. Our proof is based on [17] (and also [15]). We prove first auxiliary fact that every open cover of any metrizable space has a locally finite open refinement. We show next the main theorem that every metrizable space is paracompact. The remaining material is devoted to concepts and certain properties needed for the formulation and the proof of that theorem (see also [4]).

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The articles [19], [7], [21], [1], [20], [10], [5], [6], [13], [12], [9], [14], [4], [16], [2], [22], [3], [11], and [8] provide the notation and terminology for this paper.

1. Selected Properties of Real Numbers

In this paper r, u denote real numbers and n, k denote natural numbers.

The following propositions are true:

(3)[1] If $r > 0$ and $u > 0$, then there exists a natural number k such that $\frac{u}{2^k} \leq r$.

(4) If $k \geq n$ and $r \geq 1$, then $r^k \geq r^n$.

2. Certain Functions Defined on Families of Sets

In the sequel R denotes a binary relation and A denotes a set.

One can prove the following proposition

(5) If R well orders A, then $R|^2 A$ well orders A and $A = \mathrm{field}(R|^2 A)$.

The scheme *MinSet* deals with a set $\mathcal{A}$, a binary relation $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a set X such that $X \in \mathcal{A}$ and $\mathcal{P}[X]$ and for every set Y such that $Y \in \mathcal{A}$ and $\mathcal{P}[Y]$ holds $\langle X, Y \rangle \in \mathcal{B}$

provided the parameters satisfy the following conditions:

- $\mathcal{B}$ well orders $\mathcal{A}$, and
- There exists a set X such that $X \in \mathcal{A}$ and $\mathcal{P}[X]$.

Let F_1 be a set, let R be a binary relation, and let B be an element of F_1. The functor $\bigcup_{\beta <_R B} \beta$ is defined by:

[1] The propositions (1) and (2) have been removed.

(Def. 1) $\bigcup_{\beta <_R B} \beta = \bigcup (R\text{-Seg}(B))$.

Let F_1 be a set and let R be a binary relation. The disjoint family of F_1, R is defined as follows:

(Def. 2) $A \in$ the disjoint family of F_1, R iff there exists an element B of F_1 such that $B \in F_1$ and $A = B \setminus \bigcup_{\beta <_R B} \beta$.

Let X be a set, let n be a natural number, and let f be a function from $\mathbb{N}$ into 2^X. The functor $\bigcup_{\kappa < n} f(\kappa)$ is defined as follows:

(Def. 3) $\bigcup_{\kappa < n} f(\kappa) = \bigcup (f^{\circ}(\mathrm{Seg}\, n \setminus \{n\}))$.

3. Paracompactness of Metrizable Spaces

For simplicity, we adopt the following convention: P_1 denotes a non empty topological space, P_2 denotes a metric space, F_1, G_1, H_1 denote families of subsets of P_1, and V, W denote subsets of P_1.

One can prove the following propositions:

(6) Suppose P_1 is a T_3 space. Let given F_1. Suppose F_1 is a cover of P_1 and open. Then there exists H_1 such that H_1 is open and a cover of P_1 and for every V such that $V \in H_1$ there exists W such that $W \in F_1$ and $\overline{V} \subseteq W$.

(7) Let given P_1, F_1. Suppose P_1 is a T_2 space and paracompact and F_1 is a cover of P_1 and open. Then there exists G_1 such that G_1 is open and a cover of P_1 and $\mathrm{clf}\, G_1$ is finer than F_1 and G_1 is locally finite.

(8) Let f be a function from $[\!:$the carrier of P_1, the carrier of $P_1\!:\!]$ into $\mathbb{R}$. Suppose f is a metric of the carrier of P_1. Suppose $P_2 = \mathrm{MetrSp}(($the carrier of $P_1)$, $f)$. Then the carrier of $P_2 =$ the carrier of P_1.

(11)[2] Let f be a function from $[\!:$the carrier of P_1, the carrier of $P_1\!:\!]$ into $\mathbb{R}$. Suppose f is a metric of the carrier of P_1. Suppose $P_2 = \mathrm{MetrSp}(($the carrier of $P_1)$, $f)$. Then F_1 is a family of subsets of P_1 if and only if F_1 is a family of subsets of P_2.

In the sequel n denotes a natural number.

Let P_2 be a non empty set, let g be a function from $\mathbb{N}$ into $(2^{2^{P_2}})^*$, and let us consider n. Then $g(n)$ is a finite sequence of elements of $2^{2^{P_2}}$.

One can prove the following two propositions:

(12) Suppose P_1 is metrizable. Let F_1 be a family of subsets of P_1. Suppose F_1 is a cover of P_1 and open. Then there exists a family G_1 of subsets of P_1 which is open, a cover of P_1, finer than F_1, and locally finite.

(13) If P_1 is metrizable, then P_1 is paracompact.

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