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Order Sorted Quotient Algebra[1]

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The articles [7], [15], [20], [23], [25], [4], [24], [6], [14], [13], [18], [8], [5], [3], [1], [19], [16], [2], [11], [17], [9], [12], [22], [21], and [10] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let R be a non empty poset. Note that there exists an order sorted set of R which is binary relation yielding.

Let R be a non empty poset, let A, B be many sorted sets indexed by the carrier of R, and let I_1 be a many sorted relation between A and B. We say that I_1 is os-compatible if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let s_1, s_2 be elements of R. Suppose $s_1 \leq s_2$. Let x, y be sets. If $x \in A(s_1)$ and $y \in B(s_1)$, then $\langle x, y \rangle \in I_1(s_1)$ iff $\langle x, y \rangle \in I_1(s_2)$.

Let R be a non empty poset and let A, B be many sorted sets indexed by the carrier of R. Note that there exists a many sorted relation between A and B which is os-compatible.

Let R be a non empty poset and let A, B be many sorted sets indexed by the carrier of R. An order sorted relation of A, B is an os-compatible many sorted relation between A and B.

The following proposition is true

(1) Let R be a non empty poset, A, B be many sorted sets indexed by the carrier of R, and O_1 be a many sorted relation between A and B. If O_1 is os-compatible, then O_1 is an order sorted set of R.

Let R be a non empty poset and let A, B be many sorted sets indexed by R. One can check that every many sorted relation between A and B which is os-compatible is also order-sorted.

Let R be a non empty poset and let A be a many sorted set indexed by the carrier of R. An order sorted relation of A is an order sorted relation of A, A.

Let S be an order sorted signature and let U_1 be an order sorted algebra of S. A many sorted relation indexed by U_1 is said to be an order sorted relation of U_1 if:

(Def. 3)[1] It is os-compatible.

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[1] The definition (Def. 2) has been removed.

Let S be an order sorted signature and let U_1 be an order sorted algebra of S. One can verify that there exists an order sorted relation of U_1 which is equivalence.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Observe that there exists an equivalence order sorted relation of U_1 which is MSCongruence-like.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. An order sorted congruence of U_1 is a MSCongruence-like equivalence order sorted relation of U_1.

Let R be a non empty poset. The functor PathRel R yielding an equivalence relation of the carrier of R is defined by the condition (Def. 4).

(Def. 4) Let x, y be sets. Then $\langle x, y \rangle \in \mathrm{PathRel}\, R$ if and only if the following conditions are satisfied:

(i) $x \in$ the carrier of R,

(ii) $y \in$ the carrier of R, and

(iii) there exists a finite sequence p of elements of the carrier of R such that $1 < \mathrm{len}\, p$ and $p(1) = x$ and $p(\mathrm{len}\, p) = y$ and for every natural number n such that $2 \leq n$ and $n \leq \mathrm{len}\, p$ holds $\langle p(n), p(n-1) \rangle \in$ the internal relation of R or $\langle p(n-1), p(n) \rangle \in$ the internal relation of R.

The following proposition is true

(2) For every non empty poset R and for all elements s_1, s_2 of R such that $s_1 \leq s_2$ holds $\langle s_1, s_2 \rangle \in \mathrm{PathRel}\, R$.

Let R be a non empty poset and let s_1, s_2 be elements of R. The predicate $s_1 \cong s_2$ is defined as follows:

(Def. 5) $\langle s_1, s_2 \rangle \in \mathrm{PathRel}\, R$.

Let us notice that the predicate $s_1 \cong s_2$ is reflexive and symmetric.

Next we state the proposition

(3) For every non empty poset R and for all elements s_1, s_2, s_3 of R such that $s_1 \cong s_2$ and $s_2 \cong s_3$ holds $s_1 \cong s_3$.

Let R be a non empty poset. The functor Components R yielding a non empty family of subsets of R is defined by:

(Def. 6) $\mathrm{Components}\, R = \mathrm{Classes}\, \mathrm{PathRel}\, R$.

Let R be a non empty poset. One can verify that every element of Components R is non empty.

Let R be a non empty poset. A component of R is an element of Components R.

Let R be a non empty poset and let s_1 be an element of R. The functor $\cdot_{\mathrm{CSp}}\, s_1$ yields a component of R and is defined by:

(Def. 8)[2] $\cdot_{\mathrm{CSp}}\, s_1 = [s_1]_{\mathrm{PathRel}\, R}$.

One can prove the following two propositions:

(4) For every non empty poset R and for every element s_1 of R holds $s_1 \in \cdot_{\mathrm{CSp}}\, s_1$.

(5) For every non empty poset R and for all elements s_1, s_2 of R such that $s_1 \leq s_2$ holds $\cdot_{\mathrm{CSp}}\, s_1 = \cdot_{\mathrm{CSp}}\, s_2$.

Let R be a non empty poset, let A be a many sorted set indexed by the carrier of R, and let C be a component of R. A-carrier of C is defined by:

(Def. 9) A-carrier of $C = \bigcup \{A(s); s$ ranges over elements of R: $s \in C\}$.

The following proposition is true

[2] The definition (Def. 7) has been removed.

(6) Let R be a non empty poset, A be a many sorted set indexed by the carrier of R, s be an element of R, and x be a set. If $x \in A(s)$, then $x \in A$-carrier of $\cdot_{\rm CSp}\, s$.

Let R be a non empty poset. We say that R is locally directed if and only if:

(Def. 10) Every component of R is directed.

One can prove the following three propositions:

(7) For every discrete non empty poset R and for all elements x, y of R such that $\langle x, y\rangle \in$ PathRel R holds $x = y$.

(8) For every discrete non empty poset R and for every component C of R there exists an element x of R such that $C = \{x\}$.

(9) Every discrete non empty poset is locally directed.

Let us mention that there exists a non empty poset which is locally directed.

Let us note that there exists an order sorted signature which is locally directed.

Let us observe that every non empty poset which is discrete is also locally directed.

Let S be a locally directed non empty poset. One can verify that every component of S is directed.

Next we state the proposition

(10) $\emptyset$ is an equivalence relation of $\emptyset$.

Let S be a locally directed order sorted signature, let A be an order sorted algebra of S, let E be an equivalence order sorted relation of A, and let C be a component of S. The functor CompClass(E, C) yields an equivalence relation of (the sorts of A)-carrier of C and is defined as follows:

(Def. 11) For all sets x, y holds $\langle x, y\rangle \in$ CompClass(E, C) iff there exists an element s_1 of S such that $s_1 \in C$ and $\langle x, y\rangle \in E(s_1)$.

Let S be a locally directed order sorted signature, let A be an order sorted algebra of S, let E be an equivalence order sorted relation of A, and let s_1 be an element of S. The functor OSClass(E, s_1) yielding a subset of Classes CompClass$(E, \cdot_{\rm CSp}\, s_1)$ is defined by:

(Def. 12) For every set z holds $z \in$ OSClass(E, s_1) iff there exists a set x such that $x \in$ (the sorts of $A)(s_1)$ and $z = [x]_{{\rm CompClass}(E, \cdot_{\rm CSp}\, s_1)}$.

Let S be a locally directed order sorted signature, let A be a non-empty order sorted algebra of S, let E be an equivalence order sorted relation of A, and let s_1 be an element of S. One can check that OSClass(E, s_1) is non empty.

Next we state the proposition

(11) Let S be a locally directed order sorted signature, A be an order sorted algebra of S, E be an equivalence order sorted relation of A, and s_1, s_2 be elements of S. If $s_1 \leq s_2$, then OSClass$(E, s_1) \subseteq$ OSClass(E, s_2).

Let S be a locally directed order sorted signature, let A be an order sorted algebra of S, and let E be an equivalence order sorted relation of A. The functor OSClass E yields an order sorted set of S and is defined as follows:

(Def. 13) For every element s_1 of S holds (OSClass $E)(s_1) =$ OSClass(E, s_1).

Let S be a locally directed order sorted signature, let A be a non-empty order sorted algebra of S, and let E be an equivalence order sorted relation of A. One can verify that OSClass E is non-empty.

Let S be a locally directed order sorted signature, let U_1 be a non-empty order sorted algebra of S, let E be an equivalence order sorted relation of U_1, let s be an element of S, and let x be an element of (the sorts of $U_1)(s)$. The functor OSClass(E, x) yielding an element of OSClass(E, s) is defined as follows:

(Def. 14) $\mathrm{OSClass}(E,x) = [x]_{\mathrm{CompClass}(E,\cdot_{\mathrm{CSp}}\, s)}$.

Next we state three propositions:

(12) Let R be a locally directed non empty poset and x, y be elements of R. Given an element z of R such that $z \leq x$ and $z \leq y$. Then there exists an element u of R such that $x \leq u$ and $y \leq u$.

(13) Let S be a locally directed order sorted signature, U_1 be a non-empty order sorted algebra of S, E be an equivalence order sorted relation of U_1, s be an element of S, and x, y be elements of (the sorts of $U_1)(s)$. Then $\mathrm{OSClass}(E,x) = \mathrm{OSClass}(E,y)$ if and only if $\langle x, y\rangle \in E(s)$.

(14) Let S be a locally directed order sorted signature, U_1 be a non-empty order sorted algebra of S, E be an equivalence order sorted relation of U_1, s_1, s_2 be elements of S, and x be an element of (the sorts of $U_1)(s_1)$. Suppose $s_1 \leq s_2$. Let y be an element of (the sorts of $U_1)(s_2)$. If $y = x$, then $\mathrm{OSClass}(E,x) = \mathrm{OSClass}(E,y)$.

2. Order Sorted Quotient Algebra

In the sequel S denotes a locally directed order sorted signature and o denotes an element of the operation symbols of S.

Let us consider S, o, let A be a non-empty order sorted algebra of S, let R be an order sorted congruence of A, and let x be an element of $\mathrm{Args}(o,A)$. The functor $Rosx$ yielding an element of $\prod(\mathrm{OSClass}\,R \cdot \mathrm{Arity}(o))$ is defined by the condition (Def. 15).

(Def. 15) Let n be a natural number. Suppose $n \in \mathrm{dom}\,\mathrm{Arity}(o)$. Then there exists an element y of (the sorts of $A)(\mathrm{Arity}(o)_n)$ such that $y = x(n)$ and $(Rosx)(n) = \mathrm{OSClass}(R,y)$.

Let us consider S, o, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotRes}(R,o)$ yields a function from ((the sorts of $A) \cdot$ (the result sort of $S))(o)$ into $(\mathrm{OSClass}\,R \cdot$ the result sort of $S)(o)$ and is defined as follows:

(Def. 16) For every element x of (the sorts of $A)$(the result sort of o) holds $(\mathrm{OSQuotRes}(R,o))(x) = \mathrm{OSClass}(R,x)$.

The functor $\mathrm{OSQuotArgs}(R,o)$ yields a function from ((the sorts of $A)^{\#} \cdot$ the arity of $S)(o)$ into $((\mathrm{OSClass}\,R)^{\#} \cdot$ the arity of $S)(o)$ and is defined as follows:

(Def. 17) For every element x of $\mathrm{Args}(o,A)$ holds $(\mathrm{OSQuotArgs}(R,o))(x) = Rosx$.

Let us consider S, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotRes}\,R$ yielding a many sorted function from (the sorts of $A) \cdot$ (the result sort of S) into $\mathrm{OSClass}\,R \cdot$ the result sort of S is defined by:

(Def. 18) For every operation symbol o of S holds $(\mathrm{OSQuotRes}\,R)(o) = \mathrm{OSQuotRes}(R,o)$.

The functor $\mathrm{OSQuotArgs}\,R$ yielding a many sorted function from (the sorts of $A)^{\#} \cdot$ the arity of S into $(\mathrm{OSClass}\,R)^{\#} \cdot$ the arity of S is defined by:

(Def. 19) For every operation symbol o of S holds $(\mathrm{OSQuotArgs}\,R)(o) = \mathrm{OSQuotArgs}(R,o)$.

One can prove the following proposition

(15) Let A be a non-empty order sorted algebra of S, R be an order sorted congruence of A, and x be a set. Suppose $x \in ((\mathrm{OSClass}\,R)^{\#} \cdot$ the arity of $S)(o)$. Then there exists an element a of $\mathrm{Args}(o,A)$ such that $x = Rosa$.

Let us consider S, o, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor $\mathrm{OSQuotCharact}(R,o)$ yielding a function from $((\mathrm{OSClass}\,R)^{\#} \cdot$ the arity of $S)(o)$ into $(\mathrm{OSClass}\,R \cdot$ the result sort of $S)(o)$ is defined by:

(Def. 20) For every element a of $\mathrm{Args}(o,A)$ such that $Rosa \in ((\mathrm{OSClass}\,R)^{\#} \cdot$ the arity of $S)(o)$ holds $(\mathrm{OSQuotCharact}(R,o))(Rosa) = (\mathrm{OSQuotRes}(R,o) \cdot \mathrm{Den}(o,A))(a)$.

Let us consider S, let A be a non-empty order sorted algebra of S, and let R be an order sorted congruence of A. The functor OSQuotCharact$R$ yields a many sorted function from $(\mathrm{OSClass}\,R)^{\#}\cdot$ the arity of S into OSClass$R\cdot$ the result sort of S and is defined as follows:

(Def. 21) For every operation symbol o of S holds $(\mathrm{OSQuotCharact}\,R)(o) = \mathrm{OSQuotCharact}(R,o)$.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be an order sorted congruence of U_1. The functor QuotOSAlg(U_1,R) yields an order sorted algebra of S and is defined as follows:

(Def. 22) $\mathrm{QuotOSAlg}(U_1,R) = \langle \mathrm{OSClass}\,R, \mathrm{OSQuotCharact}\,R\rangle$.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be an order sorted congruence of U_1. One can check that QuotOSAlg(U_1,R) is strict and non-empty.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, let R be an order sorted congruence of U_1, and let s be an element of S. The functor OSNatHom(U_1,R,s) yielding a function from (the sorts of U_1)(s) into OSClass(R,s) is defined as follows:

(Def. 23) For every element x of (the sorts of U_1)(s) holds $(\mathrm{OSNatHom}(U_1,R,s))(x) = \mathrm{OSClass}(R,x)$.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be an order sorted congruence of U_1. The functor OSNatHom(U_1,R) yielding a many sorted function from U_1 into QuotOSAlg(U_1,R) is defined by:

(Def. 24) For every element s of S holds $(\mathrm{OSNatHom}(U_1,R))(s) = \mathrm{OSNatHom}(U_1,R,s)$.

One can prove the following propositions:

(16) Let U_1 be a non-empty order sorted algebra of S and R be an order sorted congruence of U_1. Then OSNatHom(U_1,R) is an epimorphism of U_1 onto QuotOSAlg(U_1,R) and OSNatHom(U_1,R) is order-sorted.

(17) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then Congruence(F) is an order sorted congruence of U_1.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, and let F be a many sorted function from U_1 into U_2. Let us assume that F is a homomorphism of U_1 into U_2 and order-sorted. The functor OSCng$F$ yields an order sorted congruence of U_1 and is defined by:

(Def. 25) $\mathrm{OSCng}\,F = \mathrm{Congruence}(F)$.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, let F be a many sorted function from U_1 into U_2, and let s be an element of S. Let us assume that F is a homomorphism of U_1 into U_2 and order-sorted. The functor OSHomQuot(F,s) yielding a function from (the sorts of QuotOSAlg$(U_1,\mathrm{OSCng}\,F)$)(s) into (the sorts of U_2)(s) is defined by:

(Def. 26) For every element x of (the sorts of U_1)(s) holds $(\mathrm{OSHomQuot}(F,s))(\mathrm{OSClass}(\mathrm{OSCng}\,F,x)) = F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, and let F be a many sorted function from U_1 into U_2. The functor OSHomQuot$F$ yields a many sorted function from QuotOSAlg$(U_1,\mathrm{OSCng}\,F)$ into U_2 and is defined as follows:

(Def. 27) For every element s of S holds $(\mathrm{OSHomQuot}\,F)(s) = \mathrm{OSHomQuot}(F,s)$.

Next we state three propositions:

(18) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then OSHomQuot$F$ is a monomorphism of QuotOSAlg$(U_1,\mathrm{OSCng}\,F)$ into U_2 and OSHomQuot$F$ is order-sorted.

(19) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2 and order-sorted. Then OSHomQuot$F$ is an isomorphism of QuotOSAlg$(U_1, \text{OSCng}F)$ and U_2.

(20) Let U_1, U_2 be non-empty order sorted algebras of S and F be a many sorted function from U_1 into U_2. Suppose F is an epimorphism of U_1 onto U_2 and order-sorted. Then QuotOSAlg$(U_1, \text{OSCng}F)$ and U_2 are isomorphic.

Let S be an order sorted signature, let U_1 be a non-empty order sorted algebra of S, and let R be an equivalence order sorted relation of U_1. We say that R is monotone if and only if the condition (Def. 28) is satisfied.

(Def. 28) Let o_1, o_2 be operation symbols of S. Suppose $o_1 \le o_2$. Let x_1 be an element of Args(o_1, U_1) and x_2 be an element of Args(o_2, U_1). Suppose that for every natural number y such that $y \in \text{dom}\, x_1$ holds $\langle x_1(y), x_2(y)\rangle \in R(\text{Arity}(o_2)_y)$. Then $\langle (\text{Den}(o_1, U_1))(x_1), (\text{Den}(o_2, U_1))(x_2)\rangle \in R(\text{the result sort of } o_2)$.

One can prove the following two propositions:

(21) Let S be an order sorted signature and U_1 be a non-empty order sorted algebra of S. Then $[\![$the sorts of U_1, the sorts of $U_1]\!]$ is an order sorted congruence of U_1.

(22) Let S be an order sorted signature, U_1 be a non-empty order sorted algebra of S, and R be an order sorted congruence of U_1. If $R = [\![$the sorts of U_1, the sorts of $U_1]\!]$, then R is monotone.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Observe that there exists an order sorted congruence of U_1 which is monotone.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Note that there exists an equivalence order sorted relation of U_1 which is monotone.

Next we state the proposition

(23) Let S be an order sorted signature and U_1 be a non-empty order sorted algebra of S. Then every monotone equivalence order sorted relation of U_1 is MSCongruence-like.

Let S be an order sorted signature and let U_1 be a non-empty order sorted algebra of S. Note that every equivalence order sorted relation of U_1 which is monotone is also MSCongruence-like.

The following proposition is true

(24) Let S be an order sorted signature and U_1 be a monotone non-empty order sorted algebra of S. Then every order sorted congruence of U_1 is monotone.

Let S be an order sorted signature and let U_1 be a monotone non-empty order sorted algebra of S. One can check that every order sorted congruence of U_1 is monotone.

Let us consider S, let U_1 be a non-empty order sorted algebra of S, and let R be a monotone order sorted congruence of U_1. Observe that QuotOSAlg(U_1, R) is monotone.

Next we state two propositions:

(25) Let given S, U_1 be a non-empty order sorted algebra of S, and R be a monotone order sorted congruence of U_1. Then QuotOSAlg(U_1, R) is a monotone order sorted algebra of S.

(26) Let U_1 be a non-empty order sorted algebra of S, U_2 be a monotone non-empty order sorted algebra of S, and F be a many sorted function from U_1 into U_2. Suppose F is a homomorphism of U_1 into U_2 and order-sorted. Then OSCng$F$ is monotone.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, let F be a many sorted function from U_1 into U_2, let R be an order sorted congruence of U_1, and let s be an element of S. Let us assume that F is a homomorphism of U_1 into U_2 and order-sorted and $R \subseteq \text{OSCng}F$. The functor OSHomQuot$(F, R, s)$ yields a function from (the sorts of QuotOSAlg(U_1, R))(s) into (the sorts of U_2)(s) and is defined as follows:

(Def. 29) For every element x of (the sorts of U_1)(s) holds $(\mathrm{OSHomQuot}(F,R,s))(\mathrm{OSClass}(R,x)) = F(s)(x)$.

Let us consider S, let U_1, U_2 be non-empty order sorted algebras of S, let F be a many sorted function from U_1 into U_2, and let R be an order sorted congruence of U_1. The functor $\mathrm{OSHomQuot}(F,R)$ yields a many sorted function from $\mathrm{QuotOSAlg}(U_1,R)$ into U_2 and is defined as follows:

(Def. 30) For every element s of S holds $(\mathrm{OSHomQuot}(F,R))(s) = \mathrm{OSHomQuot}(F,R,s)$.

We now state the proposition

(27) Let U_1, U_2 be non-empty order sorted algebras of S, F be a many sorted function from U_1 into U_2, and R be an order sorted congruence of U_1. Suppose F is a homomorphism of U_1 into U_2 and order-sorted and $R \subseteq \mathrm{OSCng}F$. Then $\mathrm{OSHomQuot}(F,R)$ is a homomorphism of $\mathrm{QuotOSAlg}(U_1,R)$ into U_2 and $\mathrm{OSHomQuot}(F,R)$ is order-sorted.

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