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# Real Normed Space

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**Summary.** We construct a real normed space $\langle V, \|.\|\rangle$, where $V$ is a real vector space and $\|.\|$ is a norm. Auxillary properties of the norm are proved. Next, we introduce a notion of sequence in the real normed space. The basic operations on sequences (addition, subtraction, multiplication by real number) are defined. We study some properties of sequences in the real normed space and the operations on them.

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The articles [11], [5], [14], [2], [12], [6], [1], [3], [4], [15], [9], [7], [8], [10], and [13] provide the notation and terminology for this paper.

We introduce normed structures which are extensions of RLS structure and are systems

$\langle$ a carrier, a zero, an addition, an external multiplication, a norm $\rangle$,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from $[:\mathbb{R}, \text{the carrier}:]$ into the carrier, and the norm is a function from the carrier into $\mathbb{R}$.

Let us note that there exists a normed structure which is non empty.

In the sequel $X$ denotes a non empty normed structure, $a$, $b$ denote real numbers, and $x$ denotes a point of $X$.

Let us consider $X$, $x$. The functor $\|x\|$ yielding a real number is defined as follows:

(Def. 1) $\|x\| = (\text{the norm of } X)(x)$.

Let $I_1$ be a non empty normed structure. We say that $I_1$ is real normed space-like if and only if:

(Def. 2) For all points $x$, $y$ of $I_1$ and for every $a$ holds $\|x\| = 0$ iff $x = 0_{(I_1)}$ and $\|a \cdot x\| = |a| \cdot \|x\|$ and $\|x+y\| \leq \|x\| + \|y\|$.

One can check that there exists a non empty normed structure which is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A real normed space is a real normed space-like real linear space-like Abelian add-associative right zeroed right complementable non empty normed structure.

We adopt the following convention: $R_1$ denotes a real normed space and $x$, $y$, $z$, $g$ denote points of $R_1$.

Next we state a number of propositions:

(5)[1] $\|0_{(R_1)}\| = 0$.

(6) $\|-x\| = \|x\|$.

[1] The propositions (1)–(4) have been removed.

 

(7) $\|x-y\| \leq \|x\|+\|y\|$.

(8) $0 \leq \|x\|$.

(9) $\|a \cdot x+b \cdot y\| \leq |a| \cdot \|x\|+|b| \cdot \|y\|$.

(10) $\|x-y\|=0$ iff $x=y$.

(11) $\|x-y\|=\|y-x\|$.

(12) $\|x\|-\|y\| \leq \|x-y\|$.

(13) $|\|x\|-\|y\|| \leq \|x-y\|$.

(14) $\|x-z\| \leq \|x-y\|+\|y-z\|$.

(15) If $x \neq y$, then $\|x-y\| \neq 0$.

Let $R_1$ be a 1-sorted structure. Sequence of $R_1$ is defined by:

(Def. 3) It is a function from $\mathbb{N}$ into the carrier of $R_1$.

Let $R_1$ be a 1-sorted structure. Observe that every sequence of $R_1$ is function-like and relation-like.

Let $R_1$ be a non empty 1-sorted structure. We see that the sequence of $R_1$ is a function from $\mathbb{N}$ into the carrier of $R_1$.

For simplicity, we follow the rules: $S$, $S_1$, $S_2$ are sequences of $R_1$, $k$, $n$, $m$ are natural numbers, $r$ is a real number, $f$ is a function, and $d$ is a set.

We now state several propositions:

(17)[2] Let $R_1$ be a non empty 1-sorted structure and $x$ be an element of $R_1$. Then $f$ is a sequence of $R_1$ if and only if $\operatorname{dom} f=\mathbb{N}$ and for every $d$ such that $d \in \mathbb{N}$ holds $f(d)$ is an element of $R_1$.

(19)[3] For every non empty 1-sorted structure $R_1$ and for every element $x$ of $R_1$ there exists a sequence $S$ of $R_1$ such that $\operatorname{rng} S=\{x\}$.

(20) Let $R_1$ be a non empty 1-sorted structure and $S$ be a sequence of $R_1$. Given an element $x$ of $R_1$ such that let given $n$. Then $S(n)=x$. Then there exists an element $x$ of $R_1$ such that $\operatorname{rng} S=\{x\}$.

(21) Let $R_1$ be a non empty 1-sorted structure and $S$ be a sequence of $R_1$. If there exists an element $x$ of $R_1$ such that $\operatorname{rng} S=\{x\}$, then for every $n$ holds $S(n)=S(n+1)$.

(22) Let $R_1$ be a non empty 1-sorted structure and $S$ be a sequence of $R_1$. If for every $n$ holds $S(n)=S(n+1)$, then for all $n$, $k$ holds $S(n)=S(n+k)$.

(23) Let $R_1$ be a non empty 1-sorted structure and $S$ be a sequence of $R_1$. If for all $n$, $k$ holds $S(n)=S(n+k)$, then for all $n$, $m$ holds $S(n)=S(m)$.

(24) Let $R_1$ be a non empty 1-sorted structure and $S$ be a sequence of $R_1$. Suppose that for all $n$, $m$ holds $S(n)=S(m)$. Then there exists an element $x$ of $R_1$ such that for every $n$ holds $S(n)=x$.

(25) There exists $S$ such that $\operatorname{rng} S=\{0_{(R_1)}\}$.

Let $R_1$ be a non empty 1-sorted structure and let $S$ be a sequence of $R_1$. Let us observe that $S$ is constant if and only if:

(Def. 4) There exists an element $x$ of $R_1$ such that for every $n$ holds $S(n)=x$.

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[2] The proposition (16) has been removed.
[3] The proposition (18) has been removed.

The following proposition is true

(27)[4] Let $R_1$ be a non empty 1-sorted structure and $S$ be a sequence of $R_1$. Then $S$ is constant if and only if there exists an element $x$ of $R_1$ such that $\operatorname{rng} S = \{x\}$.

Let $R_1$ be a non empty 1-sorted structure, let $S$ be a sequence of $R_1$, and let us consider $n$. Then $S(n)$ is an element of $R_1$.

In this article we present several logical schemes. The scheme *ExRNSSeq* deals with a real normed space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence $S$ of $\mathcal{A}$ such that for every $n$ holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The scheme *ExRLSSeq* deals with a real linear space $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a sequence $S$ of $\mathcal{A}$ such that for every $n$ holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

Let $R_1$ be a real linear space and let $S_1$, $S_2$ be sequences of $R_1$. The functor $S_1 + S_2$ yielding a sequence of $R_1$ is defined as follows:

(Def. 5) For every $n$ holds $(S_1 + S_2)(n) = S_1(n) + S_2(n)$.

Let $R_1$ be a real linear space and let $S_1$, $S_2$ be sequences of $R_1$. The functor $S_1 - S_2$ yields a sequence of $R_1$ and is defined as follows:

(Def. 6) For every $n$ holds $(S_1 - S_2)(n) = S_1(n) - S_2(n)$.

Let $R_1$ be a real linear space, let $S$ be a sequence of $R_1$, and let $x$ be an element of $R_1$. The functor $S - x$ yields a sequence of $R_1$ and is defined by:

(Def. 7) For every $n$ holds $(S - x)(n) = S(n) - x$.

Let $R_1$ be a real linear space, let $S$ be a sequence of $R_1$, and let us consider $a$. The functor $a \cdot S$ yielding a sequence of $R_1$ is defined as follows:

(Def. 8) For every $n$ holds $(a \cdot S)(n) = a \cdot S(n)$.

Let us consider $R_1$ and let us consider $S$. We say that $S$ is convergent if and only if:

(Def. 9) There exists $g$ such that for every $r$ such that $0 < r$ there exists $m$ such that for every $n$ such that $m \leq n$ holds $\|S(n) - g\| < r$.

One can prove the following propositions:

(34)[5] If $S_1$ is convergent and $S_2$ is convergent, then $S_1 + S_2$ is convergent.

(35) If $S_1$ is convergent and $S_2$ is convergent, then $S_1 - S_2$ is convergent.

(36) If $S$ is convergent, then $S - x$ is convergent.

(37) If $S$ is convergent, then $a \cdot S$ is convergent.

Let us consider $R_1$ and let us consider $S$. The functor $\|S\|$ yields a sequence of real numbers and is defined as follows:

(Def. 10) For every $n$ holds $\|S\|(n) = \|S(n)\|$.

Next we state the proposition

(39)[6] If $S$ is convergent, then $\|S\|$ is convergent.

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[4] The proposition (26) has been removed.

[5] The propositions (28)–(33) have been removed.

[6] The proposition (38) has been removed.

Let us consider $R_1$ and let us consider $S$. Let us assume that $S$ is convergent. The functor $\lim S$ yielding a point of $R_1$ is defined by:

(Def. 11) For every $r$ such that $0 < r$ there exists $m$ such that for every $n$ such that $m \leq n$ holds $\|S(n) - \lim S\| < r$.

One can prove the following propositions:

(41)[7] If $S$ is convergent and $\lim S = g$, then $\|S - g\|$ is convergent and $\lim \|S - g\| = 0$.

(42) If $S_1$ is convergent and $S_2$ is convergent, then $\lim(S_1 + S_2) = \lim S_1 + \lim S_2$.

(43) If $S_1$ is convergent and $S_2$ is convergent, then $\lim(S_1 - S_2) = \lim S_1 - \lim S_2$.

(44) If $S$ is convergent, then $\lim(S - x) = \lim S - x$.

(45) If $S$ is convergent, then $\lim(a \cdot S) = a \cdot \lim S$.

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[7] The proposition (40) has been removed.