JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Preliminaries to Circuits, II[1]

Yatsuka Nakamura
Shinshu University, Nagano

Piotr Rudnicki
University of Alberta, Edmonton

Andrzej Trybulec
Warsaw University, Białystok

Pauline N. Kawamoto
Shinshu University, Nagano

Summary. This article is the second in a series of four articles (started with [19] and continued in [18], [20]) about modelling circuits by many sorted algebras.

First, we introduce some additional terminology for many sorted signatures. The vertices of such signatures are divided into input vertices and inner vertices. A many sorted signature is called *circuit like* if each sort is a result sort of at most one operation. Next, we introduce some notions for many sorted algebras and many sorted free algebras. Free envelope of an algebra is a free algebra generated by the sorts of the algebra. Evaluation of an algebra is defined as a homomorphism from the free envelope of the algebra into the algebra. We define depth of elements of free many sorted algebras.

A many sorted signature is said to be monotonic if every finitely generated algebra over it is locally finite (finite in each sort). Monotonic signatures are used (see [18],[20]) in modelling backbones of circuits without directed cycles.

MML Identifier: MSAFREE2.

WWW: http://mizar.org/JFM/Vol6/msafree2.html

The articles [23], [12], [27], [1], [28], [10], [15], [7], [11], [21], [3], [2], [4], [5], [6], [24], [17], [25], [13], [22], [9], [8], [14], [29], [16], [26], and [19] provide the notation and terminology for this paper.

1. Many Sorted Signatures

Let S be a many sorted signature. A vertex of S is an element of S.

Let S be a non empty many sorted signature. The functor SortsWithConstants(S) yields a subset of S and is defined by:

(Def. 1) $\mathrm{SortsWithConstants}(S) = \begin{cases} \{v; v \text{ ranges over sort symbols of } S\text{: } v \text{ has constants}\}, & \text{if } S \text{is non void,} \\ \emptyset, & \text{otherwise.} \end{cases}$

Let G be a non empty many sorted signature. The functor InputVertices(G) yielding a subset of G is defined by:

(Def. 2) $\mathrm{InputVertices}(G) = (\text{the carrier of } G) \setminus \mathrm{rng}(\text{the result sort of } G)$.

The functor InnerVertices(G) yields a subset of G and is defined by:

(Def. 3) $\mathrm{InnerVertices}(G) = \mathrm{rng}(\text{the result sort of } G)$.

[1]Partial funding for this work has been provided by: Shinshu Endowment Fund for Information Science, NSERC Grant OGP9207, JSTF award 651-93-S009.

The following propositions are true:

(1) For every void non empty many sorted signature G holds InputVertices(G) = the carrier of G.

(2) Let G be a non void non empty many sorted signature and v be a vertex of G. Suppose $v \in$ InputVertices(G). Then it is not true that there exists an operation symbol o of G such that the result sort of $o = v$.

(3) For every non empty many sorted signature G holds InputVertices$(G) \cup$ InnerVertices(G) = the carrier of G.

(4) For every non empty many sorted signature G holds InputVertices(G) misses InnerVertices(G).

(5) For every non empty many sorted signature G holds SortsWithConstants$(G) \subseteq$ InnerVertices(G).

(6) For every non empty many sorted signature G holds InputVertices(G) misses SortsWithConstants(G).

Let I_1 be a non empty many sorted signature. We say that I_1 has input vertices if and only if:

(Def. 4) InputVertices$(I_1) \neq \emptyset$.

One can check that there exists a non empty many sorted signature which is non void and has input vertices.

Let G be a non empty many sorted signature with input vertices. Observe that InputVertices(G) is non empty.

Let G be a non void non empty many sorted signature. Then InnerVertices(G) is a non empty subset of G.

Let S be a non empty many sorted signature and let M_1 be a non-empty algebra over S. A many sorted set indexed by InputVertices(S) is said to be an input assignment of M_1 if:

(Def. 5) For every vertex v of S such that $v \in$ InputVertices(S) holds it$(v) \in$ (the sorts of M_1)(v).

Let S be a non empty many sorted signature. We say that S is circuit-like if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let S' be a non void non empty many sorted signature. Suppose $S' = S$. Let o_1, o_2 be operation symbols of S'. If the result sort of o_1 = the result sort of o_2, then $o_1 = o_2$.

Let us note that every non empty many sorted signature which is void is also circuit-like.

One can check that there exists a non empty many sorted signature which is non void, circuit-like, and strict.

Let I_2 be a circuit-like non void non empty many sorted signature and let v be a vertex of I_2. Let us assume that $v \in$ InnerVertices(I_2). The action at v yields an operation symbol of I_2 and is defined by:

(Def. 7) The result sort of the action at $v = v$.

2. Free Many Sorted Algebras

Next we state the proposition

(7) Let S be a non void non empty many sorted signature, A be an algebra over S, o be an operation symbol of S, and p be a finite sequence. Suppose $\operatorname{len} p = \operatorname{len} \operatorname{Arity}(o)$ and for every natural number k such that $k \in \operatorname{dom} p$ holds $p(k) \in$ (the sorts of A)$(\operatorname{Arity}(o)_k)$. Then $p \in \operatorname{Args}(o, A)$.

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor FreeEnvelope(M_1) yielding a free strict non-empty algebra over S is defined as follows:

(Def. 8) FreeEnvelope$(M_1) =$ Free(the sorts of M_1).

The following proposition is true

(8) Let S be a non void non empty many sorted signature and M_1 be a non-empty algebra over S. Then FreeGenerator(the sorts of M_1) is a free generator set of FreeEnvelope(M_1).

Let S be a non void non empty many sorted signature and let M_1 be a non-empty algebra over S. The functor Eval(M_1) yields a many sorted function from FreeEnvelope(M_1) into M_1 and is defined by the conditions (Def. 9).

(Def. 9)(i) Eval(M_1) is a homomorphism of FreeEnvelope(M_1) into M_1, and

(ii) for every sort symbol s of S and for all sets x, y such that $y \in$ FreeSort(the sorts of M_1, s) and $y =$ the root tree of $\langle x, s\rangle$ and $x \in$ (the sorts of $M_1)(s)$ holds $(\text{Eval}(M_1))(s)(y) = x$.

One can prove the following proposition

(9) Let S be a non void non empty many sorted signature and A be a non-empty algebra over S. Then the sorts of A are a generator set of A.

Let S be a non empty many sorted signature and let I_1 be an algebra over S. We say that I_1 is finitely-generated if and only if:

(Def. 10)(i) For every non void non empty many sorted signature S' such that $S' = S$ and for every algebra A over S' such that $A = I_1$ holds there exists a generator set of A which is locally-finite if S is not void,

(ii) the sorts of I_1 are locally-finite, otherwise.

Let S be a non empty many sorted signature and let I_1 be an algebra over S. We say that I_1 is locally-finite if and only if:

(Def. 11) The sorts of I_1 are locally-finite.

Let S be a non empty many sorted signature. Observe that every non-empty algebra over S which is locally-finite is also finitely-generated.

Let S be a non empty many sorted signature. The trivial algebra of S yielding a strict algebra over S is defined by:

(Def. 12) The sorts of the trivial algebra of $S =$ (the carrier of $S) \longmapsto \{0\}$.

Let S be a non empty many sorted signature. One can verify that there exists an algebra over S which is locally-finite, non-empty, and strict.

Let I_1 be a non empty many sorted signature. We say that I_1 is monotonic if and only if:

(Def. 13) Every finitely-generated non-empty algebra over I_1 is locally-finite.

Let us note that there exists a non empty many sorted signature which is non void, finite, monotonic, and circuit-like.

Next we state several propositions:

(10) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, and v be a sort symbol of S. Then every element of the sorts of Free$(X)(v)$ is a finite decorated tree.

(11) Let S be a non void non empty many sorted signature and X be a non-empty locally-finite many sorted set indexed by the carrier of S. Then Free(X) is finitely-generated.

(12) Let S be a non void non empty many sorted signature, A be a non-empty algebra over S, v be a vertex of S, and e be an element of (the sorts of FreeEnvelope(A))(v). Suppose $v \in$ InputVertices(S). Then there exists an element x of (the sorts of A)(v) such that $e =$ the root tree of $\langle x, v\rangle$.

(13) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, o be an operation symbol of S, and p be a decorated tree yielding finite sequence. Suppose $\langle o,$ the carrier of $S\rangle$-tree$(p) \in$ (the sorts of Free(X))(the result sort of o). Then $\operatorname{len} p = \operatorname{len} \operatorname{Arity}(o)$.

(14) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, o be an operation symbol of S, and p be a decorated tree yielding finite sequence. Suppose $\langle o,$ the carrier of $S\rangle$-tree$(p) \in$ (the sorts of Free(X))(the result sort of o). Let i be a natural number. If $i \in \operatorname{dom} \operatorname{Arity}(o)$, then $p(i) \in$ (the sorts of Free(X))$(\operatorname{Arity}(o)(i))$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Note that every element of (the sorts of Free(X))(v) is finite, non empty, function-like, and relation-like.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Observe that there exists an element of (the sorts of Free(X))(v) which is function-like and relation-like.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, and let v be a vertex of S. Note that every function-like relation-like element of (the sorts of Free(X))(v) is decorated tree-like.

Let I_2 be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of I_2, and let v be a vertex of I_2. Note that there exists an element of (the sorts of Free(X))(v) which is finite.

One can prove the following proposition

(15) Let S be a non void non empty many sorted signature, X be a non-empty many sorted set indexed by the carrier of S, v be a vertex of S, o be an operation symbol of S, and e be an element of (the sorts of Free(X))(v). Suppose $v \in$ InnerVertices(S) and $e(\emptyset) = \langle o,$ the carrier of $S\rangle$. Then there exists a decorated tree yielding finite sequence p such that $\operatorname{len} p = \operatorname{len} \operatorname{Arity}(o)$ and for every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) \in$ (the sorts of Free(X))$(\operatorname{Arity}(o)(i))$.

Let S be a non void non empty many sorted signature, let X be a non-empty many sorted set indexed by the carrier of S, let v be a sort symbol of S, and let e be an element of (the sorts of Free(X))(v). The functor depth(e) yielding a natural number is defined by:

(Def. 14) There exists a finite decorated tree d_1 and there exists a finite tree t such that $d_1 = e$ and $t = \operatorname{dom} d_1$ and $\operatorname{depth}(e) = \operatorname{height} t$.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/trees_1.html`.

[3] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/card_3.html`.

[4] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/trees_2.html`.

[5] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/trees_3.html`.

[6] Grzegorz Bancerek. Joining of decorated trees. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/trees_4.html`.

[7] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[8] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/dtconstr.html.

[9] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_2.html.

[10] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[11] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[12] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[13] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.

[14] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar — part I. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lang1.html.

[15] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[16] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_3.html.

[17] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[18] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/circuit1.html.

[19] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pre_circ.html.

[20] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, II. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/circuit2.html.

[21] Andrzej Nędzusiak. σ-fields and probability. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/prob_1.html.

[22] Beata Perkowska. Free many sorted universal algebra. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msafree.html.

[23] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[24] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[25] Andrzej Trybulec. Many sorted algebras. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/msualg_1.html.

[26] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[27] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[28] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[29] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

Received December 13, 1994

Published January 2, 2004