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Midpoint algebras

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Summary. In this article basic properties of midpoint algebras are proved. We define a congruence relation ≡ on bound vectors and free vectors as the equivalence classes of ≡.

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The articles [5], [4], [8], [1], [7], [6], [2], and [3] provide the notation and terminology for this paper.

We consider midpoint algebra structures as extensions of 1-sorted structure as systems
⟨ a carrier, a midpoint operation ⟩,
where the carrier is a set and the midpoint operation is a binary operation on the carrier.

Let us note that there exists a midpoint algebra structure which is non empty.

In the sequel M_1 is a non empty midpoint algebra structure and a, b are elements of M_1.

Let us consider M_1, a, b. The functor $a\,{}^{@}\,b$ yields an element of M_1 and is defined by:

(Def. 1) $a\,{}^{@}\,b = (\text{the midpoint operation of } M_1)(a, b)$.

(Def. 2) op_2 is a binary operation on $\{\emptyset\}$.

The midpoint algebra structure EX is defined by:

(Def. 3) $\mathrm{EX} = \langle \{\emptyset\}, \mathrm{op}_2 \rangle$.

Let us observe that EX is strict and non empty.

We now state several propositions:

(5)[1] The carrier of $(\mathrm{EX}) = \{\emptyset\}$.

(6) The midpoint operation of $(\mathrm{EX}) = \mathrm{op}_2$.

(7) For every element a of EX holds $a = \emptyset$.

(8) For all elements a, b of EX holds $a\,{}^{@}\,b = \mathrm{op}_2(a, b)$.

(10)[2] Let a, b, c, d be elements of EX. Then $a\,{}^{@}\,a = a$ and $a\,{}^{@}\,b = b\,{}^{@}\,a$ and $(a\,{}^{@}\,b)\,{}^{@}\,(c\,{}^{@}\,d) = a\,{}^{@}\,c\,{}^{@}\,(b\,{}^{@}\,d)$ and there exists an element x of EX such that $x\,{}^{@}\,a = b$.

Let I_1 be a non empty midpoint algebra structure. We say that I_1 is midpoint algebra-like if and only if the condition (Def. 4) is satisfied.

[1] The propositions (1)–(4) have been removed.
[2] The proposition (9) has been removed.

(Def. 4) Let a, b, c, d be elements of I_1. Then $a\,@\,a = a$ and $a\,@\,b = b\,@\,a$ and $(a\,@\,b)\,@\,(c\,@\,d) = a\,@\,c\,@\,(b\,@\,d)$ and there exists an element x of I_1 such that $x\,@\,a = b$.

One can verify that there exists a non empty midpoint algebra structure which is strict and midpoint algebra-like.

A midpoint algebra is a midpoint algebra-like non empty midpoint algebra structure.

Let M be a midpoint algebra and let a, b be elements of M. Let us notice that the functor $a\,@\,b$ is commutative.

We adopt the following convention: M denotes a midpoint algebra and a, b, c, d, a', b', c', d', x, y, x' denote elements of M.

Next we state several propositions:

(15)[3] $(a\,@\,b)\,@\,c = a\,@\,c\,@\,(b\,@\,c)$.

(16) $a\,@\,(b\,@\,c) = (a\,@\,b)\,@\,(a\,@\,c)$.

(17) If $a\,@\,b = a$, then $a = b$.

(18) If $x\,@\,a = x'\,@\,a$, then $x = x'$.

(19) If $a\,@\,x = a\,@\,x'$, then $x = x'$.

Let us consider M, a, b, c, d. The predicate $a,b \equiv c,d$ is defined as follows:

(Def. 5) $a\,@\,d = b\,@\,c$.

Next we state several propositions:

(21)[4] $a,a \equiv b,b$.

(22) If $a,b \equiv c,d$, then $c,d \equiv a,b$.

(23) If $a,a \equiv b,c$, then $b = c$.

(24) If $a,b \equiv c,c$, then $a = b$.

(25) $a,b \equiv a,b$.

(26) There exists d such that $a,b \equiv c,d$.

(27) If $a,b \equiv c,d$ and $a,b \equiv c,d'$, then $d = d'$.

(28) If $x,y \equiv a,b$ and $x,y \equiv c,d$, then $a,b \equiv c,d$.

(29) If $a,b \equiv a',b'$ and $b,c \equiv b',c'$, then $a,c \equiv a',c'$.

In the sequel p, q, r denote elements of [: the carrier of M, the carrier of M :].

Let us consider M, p. Then p_1 is an element of M.

Let us consider M, p. Then p_2 is an element of M.

Let us consider M, p, q. The predicate $p \equiv q$ is defined by:

(Def. 6) $p_1, p_2 \equiv q_1, q_2$.

Let us notice that the predicate $p \equiv q$ is reflexive and symmetric.

Let us consider M, a, b. Then $\langle a, b\rangle$ is an element of [: the carrier of M, the carrier of M :].

One can prove the following propositions:

(31)[5] If $a,b \equiv c,d$, then $\langle a, b\rangle \equiv \langle c, d\rangle$.

(32) If $\langle a, b\rangle \equiv \langle c, d\rangle$, then $a,b \equiv c,d$.

[3] The propositions (11)–(14) have been removed.

[4] The proposition (20) has been removed.

[5] The proposition (30) has been removed.

(35)[6] If $p \equiv q$ and $p \equiv r$, then $q \equiv r$.

(36) If $p \equiv r$ and $q \equiv r$, then $p \equiv q$.

(37) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(38) If $p \equiv q$, then $r \equiv p$ iff $r \equiv q$.

(39) For every p holds $\{q : q \equiv p\}$ is a non empty subset of $[\!:$the carrier of M, the carrier of $M\!:\!]$.

Let us consider M, p. The functor $p^{\smile}$ yields a subset of $[\!:$the carrier of M, the carrier of $M\!:\!]$ and is defined as follows:

(Def. 7) $p^{\smile} = \{q : q \equiv p\}$.

Let us consider M, p. Observe that $p^{\smile}$ is non empty.
Next we state several propositions:

(41)[7] For every p holds $r \in p^{\smile}$ iff $r \equiv p$.

(42) If $p \equiv q$, then $p^{\smile} = q^{\smile}$.

(43) If $p^{\smile} = q^{\smile}$, then $p \equiv q$.

(44) If $\langle a, b\rangle^{\smile} = \langle c, d\rangle^{\smile}$, then $a\,^{@}\,d = b\,^{@}\,c$.

(45) $p \in p^{\smile}$.

Let us consider M. A non empty subset of $[\!:$the carrier of M, the carrier of $M\!:\!]$ is said to be a vector of M if:

(Def. 8) There exists p such that it $= p^{\smile}$.

In the sequel u, v, w, w' are vectors of M.
Let us consider M, p. Then $p^{\smile}$ is a vector of M.
Next we state the proposition

(48)[8] There exists u such that for every p holds $p \in u$ iff $p_{\mathbf{1}} = p_{\mathbf{2}}$.

Let us consider M. The functor I_M yielding a vector of M is defined by:

(Def. 9) $\mathrm{I}_M = \{p : p_{\mathbf{1}} = p_{\mathbf{2}}\}$.

Next we state three propositions:

(50)[9] $\mathrm{I}_M = \langle b, b\rangle^{\smile}$.

(51) There exist w, p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$.

(52) Suppose there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$ and there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w' = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$. Then $w = w'$.

Let us consider M, u, v. The functor $u+v$ yields a vector of M and is defined by:

(Def. 10) There exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $u+v = \langle p_{\mathbf{1}}, q_{\mathbf{2}}\rangle^{\smile}$.

The following proposition is true

(53) There exists b such that $u = \langle a, b\rangle^{\smile}$.

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[6] The propositions (33) and (34) have been removed.
[7] The proposition (40) has been removed.
[8] The propositions (46) and (47) have been removed.
[9] The proposition (49) has been removed.

Let us consider M, a, b. The functor $\overrightarrow{[a,b]}$ yields a vector of M and is defined as follows:

(Def. 11) $\overrightarrow{[a,b]} = \langle a, b\rangle^{\smile}$.

We now state a number of propositions:

(55)[10] There exists b such that $u = \overrightarrow{[a,b]}$.

(56) If $\langle a, b\rangle \equiv \langle c, d\rangle$, then $\overrightarrow{[a,b]} = \overrightarrow{[c,d]}$.

(57) If $\overrightarrow{[a,b]} = \overrightarrow{[c,d]}$, then $a\,@\,d = b\,@\,c$.

(58) $\mathrm{I}_M = \overrightarrow{[b,b]}$.

(59) If $\overrightarrow{[a,b]} = \overrightarrow{[a,c]}$, then $b = c$.

(60) $\overrightarrow{[a,b]} + \overrightarrow{[b,c]} = \overrightarrow{[a,c]}$.

(61) $\langle a, a\,@\,b\rangle \equiv \langle a\,@\,b, b\rangle$.

(62) $\overrightarrow{[a,a\,@\,b]} + \overrightarrow{[a,a\,@\,b]} = \overrightarrow{[a,b]}$.

(63) $(u+v)+w = u+(v+w)$.

(64) $u+\mathrm{I}_M = u$.

(65) There exists v such that $u+v = \mathrm{I}_M$.

(66) $u+v = v+u$.

(67) If $u+v = u+w$, then $v = w$.

Let us consider M, u. The functor $-u$ yields a vector of M and is defined by:

(Def. 12) $u + -u = \mathrm{I}_M$.

In the sequel X is an element of $2^{[\text{: the carrier of } M, \text{ the carrier of } M :]}$.
Let us consider M. The functor setvect$M$ yielding a set is defined by:

(Def. 13) setvect$M = \{X : X$ is a vector of $M\}$.

In the sequel x is a set.
One can prove the following proposition

(71)[11] x is a vector of M iff $x \in$ setvect$M$.

Let us consider M. Observe that setvect$M$ is non empty.
In the sequel u_1, v_1, w_1, W, W_1, W_2, T denote elements of setvect$M$.
Let us consider M, u_1, v_1. The functor u_1+v_1 yields an element of setvect$M$ and is defined by:

(Def. 14) For all u, v such that $u_1 = u$ and $v_1 = v$ holds $u_1+v_1 = u+v$.

We now state two propositions:

(74)[12] $u_1+v_1 = v_1+u_1$.

(75) $(u_1+v_1)+w_1 = u_1+(v_1+w_1)$.

Let us consider M. The functor addvect$M$ yields a binary operation on setvect$M$ and is defined as follows:

[10] The proposition (54) has been removed.
[11] The propositions (68)–(70) have been removed.
[12] The propositions (72) and (73) have been removed.

(Def. 15) For all u_1, v_1 holds $(\text{addvect}\, M)(u_1, v_1) = u_1 + v_1$.

One can prove the following two propositions:

(77)[13] For every W there exists T such that $W + T = \mathrm{I}_M$.

(78) For all W, W_1, W_2 such that $W + W_1 = \mathrm{I}_M$ and $W + W_2 = \mathrm{I}_M$ holds $W_1 = W_2$.

Let us consider M. The functor complvect M yields a unary operation on setvect M and is defined by:

(Def. 16) For every W holds $W + (\text{complvect}\, M)(W) = \mathrm{I}_M$.

Let us consider M. The functor zerovect M yielding an element of setvect M is defined as follows:

(Def. 17) $\text{zerovect}\, M = \mathrm{I}_M$.

Let us consider M. The functor vectgroup M yielding a loop structure is defined as follows:

(Def. 18) $\text{vectgroup}\, M = \langle \text{setvect}\, M, \text{addvect}\, M, \text{zerovect}\, M \rangle$.

Let us consider M. One can verify that vectgroup M is strict and non empty.

One can prove the following propositions:

(82)[14] The carrier of $\text{vectgroup}\, M = \text{setvect}\, M$.

(83) The addition of $\text{vectgroup}\, M = \text{addvect}\, M$.

(85)[15] The zero of $\text{vectgroup}\, M = \text{zerovect}\, M$.

(86) vectgroup M is add-associative, right zeroed, right complementable, and Abelian.

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[13] The proposition (76) has been removed.
[14] The propositions (79)–(81) have been removed.
[15] The proposition (84) has been removed.