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Bounded Domains and Unbounded Domains

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Summary. First, notions of inside components and outside components are introduced for any subset of n-dimensional Euclid space. Next, notions of the bounded domain and the unbounded domain are defined using the above components. If the dimension is larger than 1, and if a subset is bounded, a unbounded domain of the subset coincides with an outside component (which is unique) of the subset. For a sphere in n-dimensional space, the similar fact is true for a bounded domain. In 2 dimensional space, any rectangle also has such property. We discussed relations between the Jordan property and the concept of boundary, which are necessary to find points in domains near a curve. In the last part, we gave the sufficient criterion for belonging to the left component of some clockwise oriented finite sequences.

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The articles [38], [9], [45], [32], [46], [7], [8], [3], [40], [18], [2], [1], [34], [47], [13], [20], [6], [31], [33], [17], [29], [36], [15], [4], [10], [44], [41], [35], [5], [21], [30], [37], [24], [11], [14], [26], [12], [43], [42], [16], [19], [22], [27], [23], [28], [39], and [25] provide the notation and terminology for this paper.

1. Definitions of Bounded Domain and Unbounded Domain

We use the following convention: m, n are natural numbers, r, s are real numbers, and x, y are sets.

We now state several propositions:

(1) If $r \leq 0$, then $|r| = -r$.

(2) For all n, m such that $n \leq m$ and $m \leq n+2$ holds $m = n$ or $m = n+1$ or $m = n+2$.

(3) For all n, m such that $n \leq m$ and $m \leq n+3$ holds $m = n$ or $m = n+1$ or $m = n+2$ or $m = n+3$.

(4) For all n, m such that $n \leq m$ and $m \leq n+4$ holds $m = n$ or $m = n+1$ or $m = n+2$ or $m = n+3$ or $m = n+4$.

(5) For all real numbers a, b such that $a \geq 0$ and $b \geq 0$ holds $a+b \geq 0$.

(6) For all real numbers a, b such that $a > 0$ and $b \geq 0$ holds $a+b > 0$.

(7) For every finite sequence f such that $\operatorname{rng} f = \{x, y\}$ and $\operatorname{len} f = 2$ holds $f(1) = x$ and $f(2) = y$ or $f(1) = y$ and $f(2) = x$.

(8) Let f be an increasing finite sequence of elements of $\mathbb{R}$. If $\operatorname{rng} f = \{r, s\}$ and $\operatorname{len} f = 2$ and $r \leq s$, then $f(1) = r$ and $f(2) = s$.

In the sequel p, p_1, p_2, p_3, q, q_1, q_2 denote points of $\mathcal{E}^n_{\rm T}$.
One can prove the following propositions:

(9) $(p_1+p_2)-p_3=(p_1-p_3)+p_2$.

(10) $||q||=|q|$.

(11) $||q_1|-|q_2||\leq|q_1-q_2|$.

(12) $||[r]||=|r|$.

(13) $q-0_{\mathcal{E}^n_{\rm T}}=q$ and $0_{\mathcal{E}^n_{\rm T}}-q=-q$.

(14) For every subset P of $\mathcal{E}^n_{\rm T}$ such that P is convex holds P is connected.

(15) Let G be a non empty topological space, P be a subset of G, A be a subset of G, and Q be a subset of $G\restriction A$. If $P=Q$ and P is connected, then Q is connected.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\rm T}$. We say that A is Bounded if and only if:

(Def. 2)[1] There exists a subset C of $\mathcal{E}^n$ such that $C=A$ and C is bounded.

The following proposition is true

(16) For all subsets A, B of $\mathcal{E}^n_{\rm T}$ such that B is Bounded and $A\subseteq B$ holds A is Bounded.

Let us consider n, let A be a subset of $\mathcal{E}^n_{\rm T}$, and let B be a subset of $\mathcal{E}^n_{\rm T}$. We say that B is inside component of A if and only if:

(Def. 3) B is a component of $A^{\rm c}$ and Bounded.

Let M be a non empty metric structure. Observe that there exists a subset of M which is bounded.
One can prove the following proposition

(17) Let A be a subset of $\mathcal{E}^n_{\rm T}$ and B be a subset of $\mathcal{E}^n_{\rm T}$. Then B is inside component of A if and only if there exists a subset C of $(\mathcal{E}^n_{\rm T})\restriction A^{\rm c}$ such that $C=B$ and C is a component of $(\mathcal{E}^n_{\rm T})\restriction A^{\rm c}$ and a bounded subset of $\mathcal{E}^n$.

Let us consider n, let A be a subset of $\mathcal{E}^n_{\rm T}$, and let B be a subset of $\mathcal{E}^n_{\rm T}$. We say that B is outside component of A if and only if:

(Def. 4) B is a component of $A^{\rm c}$ and B is not Bounded.

The following propositions are true:

(18) Let A be a subset of $\mathcal{E}^n_{\rm T}$ and B be a subset of $\mathcal{E}^n_{\rm T}$. Then B is outside component of A if and only if there exists a subset C of $(\mathcal{E}^n_{\rm T})\restriction A^{\rm c}$ such that $C=B$ and C is a component of $(\mathcal{E}^n_{\rm T})\restriction A^{\rm c}$ and C is not a bounded subset of $\mathcal{E}^n$.

(19) For all subsets A, B of $\mathcal{E}^n_{\rm T}$ such that B is inside component of A holds $B\subseteq A^{\rm c}$.

(20) For all subsets A, B of $\mathcal{E}^n_{\rm T}$ such that B is outside component of A holds $B\subseteq A^{\rm c}$.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\rm T}$. The functor BDDA yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 5) $\mathrm{BDD}A=\bigcup\{B; B$ ranges over subsets of $\mathcal{E}^n_{\rm T}$: B is inside component of $A\}$.

Let us consider n and let A be a subset of $\mathcal{E}^n_{\rm T}$. The functor UBDA yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 6) $\mathrm{UBD}A=\bigcup\{B; B$ ranges over subsets of $\mathcal{E}^n_{\rm T}$: B is outside component of $A\}$.

[1] The definition (Def. 1) has been removed.

The following two propositions are true:

(21) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is convex.

(22) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is connected.

Let us consider n. Observe that $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is connected.
One can prove the following propositions:

(23) $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is a component of $\mathcal{E}^n_{\mathrm{T}}$.

(24) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}\,A$ is a union of components of $(\mathcal{E}^n_{\mathrm{T}})\restriction A^{\mathrm{c}}$.

(25) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{UBD}\,A$ is a union of components of $(\mathcal{E}^n_{\mathrm{T}})\restriction A^{\mathrm{c}}$.

(26) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is inside component of A holds $B \subseteq \mathrm{BDD}\,A$.

(27) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is outside component of A holds $B \subseteq \mathrm{UBD}\,A$.

(28) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}\,A$ misses $\mathrm{UBD}\,A$.

(29) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}\,A \subseteq A^{\mathrm{c}}$.

(30) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{UBD}\,A \subseteq A^{\mathrm{c}}$.

(31) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathrm{BDD}\,A \cup \mathrm{UBD}\,A = A^{\mathrm{c}}$.

In the sequel u is a point of $\mathcal{E}^n$.
Next we state two propositions:

(32) Let G be a non empty topological space, w_1, w_2, w_3 be points of G, and h_1, h_2 be maps from $\mathbb{I}$ into G. Suppose h_1 is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and h_2 is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map h_3 from $\mathbb{I}$ into G such that h_3 is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$ and $\mathrm{rng}\,h_3 \subseteq \mathrm{rng}\,h_1 \cup \mathrm{rng}\,h_2$.

(33) For every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \mathcal{R}^n$ holds P is connected.

Let us consider n. The functor $1 * n$ yields a finite sequence of elements of $\mathbb{R}$ and is defined by:

(Def. 7) $1 * n = n \mapsto (1$ **qua** real number$)$.

Let us consider n. Then $1 * n$ is an element of $\mathcal{R}^n$.
Let us consider n. The functor $1.\mathrm{REAL}\,n$ yielding a point of $\mathcal{E}^n_{\mathrm{T}}$ is defined as follows:

(Def. 8) $1.\mathrm{REAL}\,n = 1 * n$.

One can prove the following propositions:

(34) $|1 * n| = n \mapsto (1$ **qua** real number$)$.

(35) $|1 * n| = \sqrt{n}$.

(36) $1.\mathrm{REAL}\,1 = \langle (1$ **qua** real number$)\rangle$.

(37) $|1.\mathrm{REAL}\,n| = \sqrt{n}$.

(38) If $1 \leq n$, then $1 \leq |1.\mathrm{REAL}\,n|$.

(39) For every subset W of $\mathcal{E}^n$ such that $n \geq 1$ and $W = \mathcal{R}^n$ holds W is not bounded.

(40) Let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Then A is Bounded if and only if there exists a real number r such that for every point q of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in A$ holds $|q| < r$.

(41) If $n \geq 1$, then $\Omega_{\mathcal{E}^n_{\mathrm{T}}}$ is not Bounded.

(42) If $n \geq 1$, then $\mathrm{UBD}\,\emptyset_{\mathcal{E}^n_{\mathrm{T}}} = \mathcal{R}^n$.

(43) Let w_1, w_2, w_3 be points of $\mathcal{E}^n_{\mathrm{T}}$, P be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$, and h_1, h_2 be maps from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$. Suppose h_1 is continuous and $w_1 = h_1(0)$ and $w_2 = h_1(1)$ and h_2 is continuous and $w_2 = h_2(0)$ and $w_3 = h_2(1)$. Then there exists a map h_3 from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that h_3 is continuous and $w_1 = h_3(0)$ and $w_3 = h_3(1)$.

(44) Let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and w_1, w_2, w_3 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $\mathcal{L}(w_1, w_2) \subseteq P$ and $\mathcal{L}(w_2, w_3) \subseteq P$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that h is continuous and $w_1 = h(0)$ and $w_3 = h(1)$.

(45) Let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and w_1, w_2, w_3, w_4 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $w_4 \in P$ and $\mathcal{L}(w_1, w_2) \subseteq P$ and $\mathcal{L}(w_2, w_3) \subseteq P$ and $\mathcal{L}(w_3, w_4) \subseteq P$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that h is continuous and $w_1 = h(0)$ and $w_4 = h(1)$.

(46) Let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and w_1, w_2, w_3, w_4, w_5, w_6, w_7 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $w_1 \in P$ and $w_2 \in P$ and $w_3 \in P$ and $w_4 \in P$ and $w_5 \in P$ and $w_6 \in P$ and $w_7 \in P$ and $\mathcal{L}(w_1, w_2) \subseteq P$ and $\mathcal{L}(w_2, w_3) \subseteq P$ and $\mathcal{L}(w_3, w_4) \subseteq P$ and $\mathcal{L}(w_4, w_5) \subseteq P$ and $\mathcal{L}(w_5, w_6) \subseteq P$ and $\mathcal{L}(w_6, w_7) \subseteq P$. Then there exists a map h from $\mathbb{I}$ into $(\mathcal{E}^n_{\mathrm{T}})\restriction P$ such that h is continuous and $w_1 = h(0)$ and $w_7 = h(1)$.

(47) For all points w_1, w_2 of $\mathcal{E}^n_{\mathrm{T}}$ such that it is not true that there exists a real number r such that $w_1 = r \cdot w_2$ or $w_2 = r \cdot w_1$ holds $0_{\mathcal{E}^n_{\mathrm{T}}} \notin \mathcal{L}(w_1, w_2)$.

(48) Let w_1, w_2 be points of $\mathcal{E}^n_{\mathrm{T}}$ and P be a subset of $(\mathcal{E}^n)_{\mathrm{top}}$. Suppose $P = \mathcal{L}(w_1, w_2)$ and $0_{\mathcal{E}^n_{\mathrm{T}}} \notin \mathcal{L}(w_1, w_2)$. Then there exists a point w_0 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_0 \in \mathcal{L}(w_1, w_2)$ and $|w_0| > 0$ and $|w_0| = (\mathrm{dist}_{\min}(P))(0_{\mathcal{E}^n_{\mathrm{T}}})$.

(49) Let a be a real number, Q be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and w_1, w_4 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $Q = \{q : |q| > a\}$ and $w_1 \in Q$ and $w_4 \in Q$ and it is not true that there exists a real number r such that $w_1 = r \cdot w_4$ or $w_4 = r \cdot w_1$. Then there exist points w_2, w_3 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$.

(50) Let a be a real number, Q be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and w_1, w_4 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $Q = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $w_1 \in Q$ and $w_4 \in Q$ and it is not true that there exists a real number r such that $w_1 = r \cdot w_4$ or $w_4 = r \cdot w_1$. Then there exist points w_2, w_3 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$.

(52)[2] Every finite sequence f of elements of $\mathbb{R}$ is an element of $\mathcal{R}^{\mathrm{len}\, f}$ and a point of $\mathcal{E}^{\mathrm{len}\, f}_{\mathrm{T}}$.

(53) Let x be an element of $\mathcal{R}^n$, f, g be finite sequences of elements of $\mathbb{R}$, and r be a real number. Suppose $f = x$ and $g = r \cdot x$. Then $\mathrm{len}\, f = \mathrm{len}\, g$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, f$ holds $g_i = r \cdot f_i$.

(54) Let x be an element of $\mathcal{R}^n$ and f be a finite sequence. Suppose $x \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $x = f$. Then there exists a natural number i such that $1 \leq i$ and $i \leq n$ and $f(i) \neq 0$.

(55) Let x be an element of $\mathcal{R}^n$. Suppose $n \geq 2$ and $x \neq \langle \underbrace{0, \ldots, 0}_{n} \rangle$. Then it is not true that there exists an element y of $\mathcal{R}^n$ and there exists a real number r such that $y = r \cdot x$ or $x = r \cdot y$.

(56) Let a be a real number, Q be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and w_1, w_7 be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $n \geq 2$ and $Q = \{q : |q| > a\}$ and $w_1 \in Q$ and $w_7 \in Q$ and there exists a real number r such that $w_1 = r \cdot w_7$ or $w_7 = r \cdot w_1$. Then there exist points w_2, w_3, w_4, w_5, w_6 of $\mathcal{E}^n_{\mathrm{T}}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $w_4 \in Q$ and $w_5 \in Q$ and $w_6 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$ and $\mathcal{L}(w_4, w_5) \subseteq Q$ and $\mathcal{L}(w_5, w_6) \subseteq Q$ and $\mathcal{L}(w_6, w_7) \subseteq Q$.

[2] The proposition (51) has been removed.

(57) Let a be a real number, Q be a subset of $\mathcal{E}^n_{\rm T}$, and w_1, w_7 be points of $\mathcal{E}^n_{\rm T}$. Suppose $n \geq 2$ and $Q = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $w_1 \in Q$ and $w_7 \in Q$ and there exists a real number r such that $w_1 = r \cdot w_7$ or $w_7 = r \cdot w_1$. Then there exist points w_2, w_3, w_4, w_5, w_6 of $\mathcal{E}^n_{\rm T}$ such that $w_2 \in Q$ and $w_3 \in Q$ and $w_4 \in Q$ and $w_5 \in Q$ and $w_6 \in Q$ and $\mathcal{L}(w_1, w_2) \subseteq Q$ and $\mathcal{L}(w_2, w_3) \subseteq Q$ and $\mathcal{L}(w_3, w_4) \subseteq Q$ and $\mathcal{L}(w_4, w_5) \subseteq Q$ and $\mathcal{L}(w_5, w_6) \subseteq Q$ and $\mathcal{L}(w_6, w_7) \subseteq Q$.

(58) For every real number a such that $n \geq 1$ holds $\{q : |q| > a\} \neq \emptyset$.

(59) For every real number a and for every subset P of $\mathcal{E}^n_{\rm T}$ such that $n \geq 2$ and $P = \{q : |q| > a\}$ holds P is connected.

(60) For every real number a such that $n \geq 1$ holds $\mathcal{R}^n \setminus \{q : |q| < a\} \neq \emptyset$.

(61) For every real number a and for every subset P of $\mathcal{E}^n_{\rm T}$ such that $n \geq 2$ and $P = \mathcal{R}^n \setminus \{q : |q| < a\}$ holds P is connected.

(62) Let a be a real number, n be a natural number, and P be a subset of $\mathcal{E}^n_{\rm T}$. If $n \geq 1$ and $P = \mathcal{R}^n \setminus \{q; q$ ranges over points of $\mathcal{E}^n_{\rm T}$: $|q| < a\}$, then P is not Bounded.

(63) Let a be a real number and P be a subset of $\mathcal{E}^1_{\rm T}$. If $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\rm T}$: $\bigvee_r (q = \langle r \rangle \wedge r > a)\}$, then P is convex.

(64) Let a be a real number and P be a subset of $\mathcal{E}^1_{\rm T}$. If $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\rm T}$: $\bigvee_r (q = \langle r \rangle \wedge r < -a)\}$, then P is convex.

(65) Let a be a real number and P be a subset of $\mathcal{E}^1_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\rm T}$: $\bigvee_r (q = \langle r \rangle \wedge r > a)\}$. Then P is connected.

(66) Let a be a real number and P be a subset of $\mathcal{E}^1_{\rm T}$. Suppose $P = \{q; q$ ranges over points of $\mathcal{E}^1_{\rm T}$: $\bigvee_r (q = \langle r \rangle \wedge r < -a)\}$. Then P is connected.

(67) Let W be a subset of $\mathcal{E}^1$, a be a real number, and P be a subset of $\mathcal{E}^1_{\rm T}$. Suppose $W = \{q; q$ ranges over points of $\mathcal{E}^1_{\rm T}$: $\bigvee_r (q = \langle r \rangle \wedge r > a)\}$ and $P = W$. Then P is connected and W is not bounded.

(68) Let W be a subset of $\mathcal{E}^1$, a be a real number, and P be a subset of $\mathcal{E}^1_{\rm T}$. Suppose $W = \{q; q$ ranges over points of $\mathcal{E}^1_{\rm T}$: $\bigvee_r (q = \langle r \rangle \wedge r < -a)\}$ and $P = W$. Then P is connected and W is not bounded.

(69) Let W be a subset of $\mathcal{E}^n$, a be a real number, and P be a subset of $\mathcal{E}^n_{\rm T}$. If $n \geq 2$ and $W = \{q : |q| > a\}$ and $P = W$, then P is connected and W is not bounded.

(70) Let W be a subset of $\mathcal{E}^n$, a be a real number, and P be a subset of $\mathcal{E}^n_{\rm T}$. If $n \geq 2$ and $W = \mathcal{R}^n \setminus \{q : |q| < a\}$ and $P = W$, then P is connected and W is not bounded.

(71) Let P, P_1 be subsets of $\mathcal{E}^n_{\rm T}$, Q be a subset of $\mathcal{E}^n_{\rm T}$, and W be a subset of $\mathcal{E}^n$. Suppose $P = W$ and P is connected and W is not bounded and $P_1 = \text{Component}(\text{Down}(P, Q^{\rm c}))$ and W misses Q. Then P_1 is outside component of Q.

(72) Let A be a subset of $\mathcal{E}^n$, B be a non empty subset of $\mathcal{E}^n$, and C be a subset of $\mathcal{E}^n \restriction B$. If $A \subseteq B$ and $A = C$ and C is bounded, then A is bounded.

(73) For every subset A of $\mathcal{E}^n_{\rm T}$ such that A is compact holds A is Bounded.

(74) For every subset A of $\mathcal{E}^n_{\rm T}$ such that $1 \leq n$ and A is Bounded holds $A^{\rm c} \neq \emptyset$.

(75) Let r be a real number. Then there exists a subset B of $\mathcal{E}^n$ such that $B = \{q : |q| < r\}$ and for every subset A of $\mathcal{E}^n$ such that $A = \{q_1 : |q_1| < r\}$ holds A is bounded.

(76) Let A be a subset of $\mathcal{E}^n_{\rm T}$. Suppose $n \geq 2$ and A is Bounded. Then there exists a subset B of $\mathcal{E}^n_{\rm T}$ such that B is outside component of A and $B = \text{UBD}\,A$.

(77) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \{q : |q| < a\}$ holds P is convex.

(78) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \mathrm{Ball}(u,a)$ holds P is convex.

(79) For every real number a and for every subset P of $\mathcal{E}^n_{\mathrm{T}}$ such that $P = \{q : |q| < a\}$ holds P is connected.

In the sequel R denotes a subset of $\mathcal{E}^n_{\mathrm{T}}$ and P denotes a subset of $\mathcal{E}^n_{\mathrm{T}}$.
Next we state a number of propositions:

(80) Suppose $p \neq q$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$. Then there exists a map h from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that h is continuous and $h(0) = p$ and $h(1) = q$ and $\mathrm{rng}\, h \subseteq \mathrm{Ball}(u,r)$.

(81) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is continuous and $f(0) = p_1$ and $f(1) = p_2$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$. Then there exists a map h from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that h is continuous and $h(0) = p_1$ and $h(1) = p$ and $\mathrm{rng}\, h \subseteq \mathrm{rng}\, f \cup \mathrm{Ball}(u,r)$.

(82) Let f be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is continuous and $\mathrm{rng}\, f \subseteq P$ and $f(0) = p_1$ and $f(1) = p_2$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$ and $\mathrm{Ball}(u,r) \subseteq P$. Then there exists a map f_1 from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that f_1 is continuous and $\mathrm{rng}\, f_1 \subseteq P$ and $f_1(0) = p_1$ and $f_1(1) = p$.

(83) Let given p and P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that

(i) R is connected and open, and

(ii) $P = \{q : q \neq p \wedge q \in R \wedge \neg \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$.

Then P is open.

(84) Let P be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that

(i) R is connected and open,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$.

Then P is open.

(85) Let R be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $p \in R$ and $P = \{q : q = p \vee \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$. Then $P \subseteq R$.

(86) Let R be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. Suppose that

(i) R is connected and open,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{f : \text{map from } \mathbb{I} \text{ into } \mathcal{E}^n_{\mathrm{T}}} (f \text{is continuous} \wedge \mathrm{rng}\, f \subseteq R \wedge f(0) = p \wedge f(1) = q)\}$.

Then $R \subseteq P$.

(87) Let R be a subset of $\mathcal{E}^n_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. Suppose R is connected and open and $p \in R$ and $q \in R$ and $p \neq q$. Then there exists a map f from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$ such that f is continuous and $\mathrm{rng}\, f \subseteq R$ and $f(0) = p$ and $f(1) = q$.

(88) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number a such that $A = \{q : |q| = a\}$ holds A^{c} is open and A is closed.

(89) For every non empty subset B of $\mathcal{E}^n_{\mathrm{T}}$ such that B is open holds $(\mathcal{E}^n_{\mathrm{T}}) \restriction B$ is locally connected.

(90) Let B be a non empty subset of $\mathcal{E}^n_{\mathrm{T}}$, A be a subset of $\mathcal{E}^n_{\mathrm{T}}$, and a be a real number. If $A = \{q : |q| = a\}$ and $A^{\mathrm{c}} = B$, then $(\mathcal{E}^n_{\mathrm{T}}) \restriction B$ is locally connected.

(91) For every map f from $\mathcal{E}^n_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every q holds $f(q) = |q|$ holds f is continuous.

(92) There exists a map f from $\mathcal{E}^n_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every q holds $f(q) = |q|$ and f is continuous.

Let X, Y be non empty 1-sorted structures, let f be a map from X into Y, and let x be a set. Let us assume that x is a point of X. The functor $\pi_x f$ yields a point of Y and is defined as follows:

(Def. 10)[3] $\pi_x f = f(x)$.

The following four propositions are true:

(93) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$. Suppose g is continuous. Then there exists a map f from $\mathbb{I}$ into $\mathbb{R}^{\mathbf{1}}$ such that for every point t of $\mathbb{I}$ holds $f(t) = |g(t)|$ and f is continuous.

(94) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ and a be a real number. Suppose g is continuous and $|\pi_0 g| \leq a$ and $a \leq |\pi_1 g|$. Then there exists a point s of $\mathbb{I}$ such that $|\pi_s g| = a$.

(95) If $q = \langle r \rangle$, then $|q| = |r|$.

(96) Let A be a subset of $\mathcal{E}^n_{\rm T}$ and a be a real number. Suppose $n \geq 1$ and $a > 0$ and $A = \{q : |q| = a\}$. Then there exists a subset B of $\mathcal{E}^n_{\rm T}$ such that B is inside component of A and $B = {\rm BDD}\,A$.

2. Bounded and Unbounded Domains of Rectangles

In the sequel D denotes a non vertical non horizontal non empty compact subset of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(97) len the Go-board of SpStSeq$D = 2$ and width the Go-board of SpStSeq$D = 2$ and $({\rm SpStSeq}\,D)_1 =$ the Go-board of SpStSeq$D \circ (1,2)$ and $({\rm SpStSeq}\,D)_2 =$ the Go-board of SpStSeq$D \circ (2,2)$ and $({\rm SpStSeq}\,D)_3 =$ the Go-board of SpStSeq$D \circ (2,1)$ and $({\rm SpStSeq}\,D)_4 =$ the Go-board of SpStSeq$D \circ (1,1)$ and $({\rm SpStSeq}\,D)_5 =$ the Go-board of SpStSeq$D \circ (1,2)$.

(98) LeftComp(SpStSeq$D$) is non Bounded.

(99) ${\rm LeftComp}({\rm SpStSeq}\,D) \subseteq {\rm UBD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$.

(100) Let G be a topological space and A, B, C be subsets of G. Suppose A is a component of G and B is a component of G and C is connected and A meets C and B meets C. Then $A = B$.

(101) For every subset B of $\mathcal{E}^2_{\rm T}$ such that B is a component of $(\widetilde{\mathcal{L}}({\rm SpStSeq}\,D))^{\rm c}$ and B is not Bounded holds $B = {\rm LeftComp}({\rm SpStSeq}\,D)$.

(102) ${\rm RightComp}({\rm SpStSeq}\,D) \subseteq {\rm BDD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$ and RightComp(SpStSeq$D$) is Bounded.

(103) ${\rm LeftComp}({\rm SpStSeq}\,D) = {\rm UBD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$ and ${\rm RightComp}({\rm SpStSeq}\,D) = {\rm BDD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$.

(104) ${\rm UBD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D) \neq \emptyset$ and ${\rm UBD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$ is outside component of $\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$ and ${\rm BDD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D) \neq \emptyset$ and ${\rm BDD}\,\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$ is inside component of $\widetilde{\mathcal{L}}({\rm SpStSeq}\,D)$.

[3] The definition (Def. 9) has been removed.

3. Jordan Property and Boundary Property

We now state several propositions:

(105) Let G be a non empty topological space and A be a subset of G. Suppose $A^{\rm c} \neq \emptyset$. Then A is boundary if and only if for every set x and for every subset V of G such that $x \in A$ and $x \in V$ and V is open there exists a subset B of G such that B is a component of $A^{\rm c}$ and V meets B.

(106) Let A be a subset of $\mathcal{E}^2_{\rm T}$. Suppose $A^{\rm c} \neq \emptyset$. Then A is boundary and Jordan if and only if there exist subsets A_1, A_2 of $\mathcal{E}^2_{\rm T}$ such that $A^{\rm c} = A_1 \cup A_2$ and A_1 misses A_2 and $\overline{A_1} \setminus A_1 = \overline{A_2} \setminus A_2$ and $A = \overline{A_1} \setminus A_1$ and for all subsets C_1, C_2 of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$ such that $C_1 = A_1$ and $C_2 = A_2$ holds C_1 is a component of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$ and C_2 is a component of $(\mathcal{E}^2_{\rm T}) \restriction A^{\rm c}$.

(107) For every point p of $\mathcal{E}^n_{\rm T}$ and for every subset P of $\mathcal{E}^n_{\rm T}$ such that $n \geq 1$ and $P = \{p\}$ holds P is boundary.

(108) For all points p, q of $\mathcal{E}^2_{\rm T}$ and for every r such that $p_{\mathbf{1}} = q_{\mathbf{2}}$ and $-p_{\mathbf{2}} = q_{\mathbf{1}}$ and $p = r \cdot q$ holds $p_{\mathbf{1}} = 0$ and $p_{\mathbf{2}} = 0$ and $p = 0_{\mathcal{E}^2_{\rm T}}$.

(109) For all points q_1, q_2 of $\mathcal{E}^2_{\rm T}$ holds $\mathcal{L}(q_1, q_2)$ is boundary.

Let q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. One can verify that $\mathcal{L}(q_1, q_2)$ is boundary.
One can prove the following proposition

(110) For every finite sequence f of elements of $\mathcal{E}^2_{\rm T}$ holds $\widetilde{\mathcal{L}}(f)$ is boundary.

Let f be a finite sequence of elements of $\mathcal{E}^2_{\rm T}$. Note that $\widetilde{\mathcal{L}}(f)$ is boundary.
Next we state several propositions:

(111) For every point e_1 of $\mathcal{E}^n$ and for all points p, q of $\mathcal{E}^n_{\rm T}$ such that $p = e_1$ and $q \in \mathrm{Ball}(e_1, r)$ holds $|p - q| < r$ and $|q - p| < r$.

(112) Let a be a real number and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $a > 0$ and $p \in \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in \mathrm{UBD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $|p - q| < a$.

(113) $\mathcal{R}^0 = \{0_{\mathcal{E}^0_{\rm T}}\}$.

(114) For every subset A of $\mathcal{E}^n_{\rm T}$ such that A is Bounded holds $\mathrm{BDD}\, A$ is Bounded.

(115) Let G be a non empty topological space and A, B, C, D be subsets of G. Suppose A is a component of G and B is a component of G and C is a component of G and $A \cup B =$ the carrier of G and C misses A. Then $C = B$.

(116) For every subset A of $\mathcal{E}^2_{\rm T}$ such that A is Bounded and Jordan holds $\mathrm{BDD}\, A$ is inside component of A.

(117) Let a be a real number and p be a point of $\mathcal{E}^2_{\rm T}$. Suppose $a > 0$ and $p \in \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in \mathrm{BDD}\, \widetilde{\mathcal{L}}(\mathrm{SpStSeq}\, D)$ and $|p - q| < a$.

4. Points in LeftComp

In the sequel f is a clockwise oriented non constant standard special circular sequence.
One can prove the following propositions:

(118) For every point p of $\mathcal{E}^2_{\rm T}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{1}} < \text{W-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

(119) For every point p of $\mathcal{E}^2_{\rm T}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{1}} > \text{E-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

(120) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{2}} < \text{S-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

(121) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $f_1 = \mathrm{N}_{\min}(\widetilde{\mathcal{L}}(f))$ and $p_{\mathbf{2}} > \text{N-bound}(\widetilde{\mathcal{L}}(f))$ holds $p \in \mathrm{LeftComp}(f)$.

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