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Isomorphisms of Categories

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Summary. We continue the development of the category theory basically following [8] (compare also [7]). We define the concept of isomorphic categories and prove basic facts related, e.g. that the Cartesian product of categories is associative up to the isomorphism. We introduce the composition of a functor and a transformation, and of transformation and a functor, and afterwards we define again those concepts for natural transformations. Let us observe, that we have to duplicate those concepts because of the permissiveness: if a functor F is not naturally transformable to G, then natural transformation from F to G has no fixed meaning, hence we cannot claim that the composition of it with a functor as a transformation results in a natural transformation. We define also the so called horizontal composition of transformations ([8], p. 140, exercise **4**.**2**,**5**(C)) and prove *interchange law* ([7], p.44). We conclude with the definition of equivalent categories.

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The articles [9], [5], [11], [2], [3], [1], [4], [6], and [10] provide the notation and terminology for this paper.

We use the following convention: A, B, C, D denote categories, F denotes a functor from A to B, and G denotes a functor from B to C.

One can prove the following propositions:

(1) For all functions F, G such that F is one-to-one and G is one-to-one holds $[: F, G :]$ is one-to-one.

(2) $\operatorname{rng}\pi_1(A \times B) =$ the morphisms of A and $\operatorname{rng}\pi_2(B \times A) =$ the morphisms of A.

(3) For every morphism f of A such that f is invertible holds $F(f)$ is invertible.

(4) For every functor F from A to B and for every functor G from B to A holds $F \cdot \mathrm{id}_A = F$ and $\mathrm{id}_A \cdot G = G$.

(7)[1] Let F_1, F_2 be functors from A to B. Suppose F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2 and a be an object of A. Then $t(a) \in \hom(F_1(a), F_2(a))$.

(8) Let F_1, F_2 be functors from A to B and G_1, G_2 be functors from B to C. Suppose F_1 is transformable to F_2 and G_1 is transformable to G_2. Then $G_1 \cdot F_1$ is transformable to $G_2 \cdot F_2$.

(9) Let F_1, F_2 be functors from A to B. Suppose F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2. Suppose t is invertible. Let a be an object of A. Then $F_1(a)$ and $F_2(a)$ are isomorphic.

[1] The propositions (5) and (6) have been removed.

Let us consider C, D. Let us note that the functor from C to D can be characterized by the following (equivalent) condition:

(Def. 1)(i) For every object c of C there exists an object d of D such that $\mathrm{it}(\mathrm{id}_c) = \mathrm{id}_d$,

(ii) for every morphism f of C holds $\mathrm{it}(\mathrm{id}_{\mathrm{dom}\, f}) = \mathrm{id}_{\mathrm{dom}\, \mathrm{it}(f)}$ and $\mathrm{it}(\mathrm{id}_{\mathrm{cod}\, f}) = \mathrm{id}_{\mathrm{cod}\, \mathrm{it}(f)}$, and

(iii) for all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $\mathrm{it}(g \cdot f) = \mathrm{it}(g) \cdot \mathrm{it}(f)$.

Let us consider A. Then id_A is a functor from A to A. Let us consider B, C, let F be a functor from A to B, and let G be a functor from B to C. Then $G \cdot F$ is a functor from A to C.

In the sequel o, m are sets.

We now state three propositions:

(10) If F is an isomorphism, then for every morphism g of B there exists a morphism f of A such that $F(f) = g$.

(11) If F is an isomorphism, then for every object b of B there exists an object a of A such that $F(a) = b$.

(12) If F is one-to-one, then $\mathrm{Obj}\, F$ is one-to-one.

Let us consider A, B, F. Let us assume that F is an isomorphism. The functor F^{-1} yielding a functor from B to A is defined as follows:

(Def. 2) $F^{-1} = F^{-1}$.

Let us consider A, B, F. Let us observe that F is isomorphic if and only if:

(Def. 3) F is one-to-one and $\mathrm{rng}\, F =$ the morphisms of B.

We introduce F is an isomorphism as a synonym of F is isomorphic.

Next we state several propositions:

(13) If F is an isomorphism, then F^{-1} is an isomorphism.

(14) If F is an isomorphism, then $(\mathrm{Obj}\, F)^{-1} = \mathrm{Obj}(F^{-1})$.

(15) If F is an isomorphism, then $(F^{-1})^{-1} = F$.

(16) If F is an isomorphism, then $F \cdot F^{-1} = \mathrm{id}_B$ and $F^{-1} \cdot F = \mathrm{id}_A$.

(17) If F is an isomorphism and G is an isomorphism, then $G \cdot F$ is an isomorphism.

Let us consider A, B. We say that A and B are isomorphic if and only if:

(Def. 4) There exists a functor from A to B which is an isomorphism.

Let us notice that the predicate A and B are isomorphic is reflexive and symmetric. We introduce $A \cong B$ as a synonym of A and B are isomorphic.

One can prove the following propositions:

(20)[2] If $A \cong B$ and $B \cong C$, then $A \cong C$.

(21) $[: \dot{\circlearrowleft}(o, m), A :] \cong A$.

(22) $[: A, B :] \cong [: B, A :]$.

(23) $[: [: A, B :], C :] \cong [: A, [: B, C :] :]$.

(24) If $A \cong B$ and $C \cong D$, then $[: A, C :] \cong [: B, D :]$.

[2] The propositions (18) and (19) have been removed.

Let us consider A, B, C and let F_1, F_2 be functors from A to B. Let us assume that F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2 and let G be a functor from B to C. The functor $G \cdot t$ yielding a transformation from $G \cdot F_1$ to $G \cdot F_2$ is defined by:

(Def. 5) $G \cdot t = G \cdot t$.

Let us consider A, B, C and let G_1, G_2 be functors from B to C. Let us assume that G_1 is transformable to G_2. Let F be a functor from A to B and let t be a transformation from G_1 to G_2. The functor $t \cdot F$ yields a transformation from $G_1 \cdot F$ to $G_2 \cdot F$ and is defined by:

(Def. 6) $t \cdot F = t \cdot \mathrm{Obj}\, F$.

We now state three propositions:

(25) Let G_1, G_2 be functors from B to C. Suppose G_1 is transformable to G_2. Let F be a functor from A to B, t be a transformation from G_1 to G_2, and a be an object of A. Then $(t \cdot F)(a) = t(F(a))$.

(26) Let F_1, F_2 be functors from A to B. Suppose F_1 is transformable to F_2. Let t be a transformation from F_1 to F_2, G be a functor from B to C, and a be an object of A. Then $(G \cdot t)(a) = G(t(a))$.

(27) Let F_1, F_2 be functors from A to B and G_1, G_2 be functors from B to C. Suppose F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2. Then $G_1 \cdot F_1$ is naturally transformable to $G_2 \cdot F_2$.

Let us consider A, B, C and let F_1, F_2 be functors from A to B. Let us assume that F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2 and let G be a functor from B to C. The functor $G \cdot t$ yields a natural transformation from $G \cdot F_1$ to $G \cdot F_2$ and is defined as follows:

(Def. 7) $G \cdot t = G \cdot t$.

The following proposition is true

(28) Let F_1, F_2 be functors from A to B. Suppose F_1 is naturally transformable to F_2. Let t be a natural transformation from F_1 to F_2, G be a functor from B to C, and a be an object of A. Then $(G \cdot t)(a) = G(t(a))$.

Let us consider A, B, C and let G_1, G_2 be functors from B to C. Let us assume that G_1 is naturally transformable to G_2. Let F be a functor from A to B and let t be a natural transformation from G_1 to G_2. The functor $t \cdot F$ yielding a natural transformation from $G_1 \cdot F$ to $G_2 \cdot F$ is defined as follows:

(Def. 8) $t \cdot F = t \cdot F$.

We now state the proposition

(29) Let G_1, G_2 be functors from B to C. Suppose G_1 is naturally transformable to G_2. Let F be a functor from A to B, t be a natural transformation from G_1 to G_2, and a be an object of A. Then $(t \cdot F)(a) = t(F(a))$.

For simplicity, we adopt the following convention: F, F_1, F_2, F_3 are functors from A to B, G, G_1, G_2, G_3 are functors from B to C, H, H_1, H_2 are functors from C to D, s is a natural transformation from F_1 to F_2, s' is a natural transformation from F_2 to F_3, t is a natural transformation from G_1 to G_2, t' is a natural transformation from G_2 to G_3, and u is a natural transformation from H_1 to H_2.

We now state a number of propositions:

(30) If F_1 is naturally transformable to F_2, then for every object a of A holds $\mathrm{hom}(F_1(a), F_2(a)) \neq \emptyset$.

(31) Suppose F_1 is naturally transformable to F_2. Let t_1, t_2 be natural transformations from F_1 to F_2. If for every object a of A holds $t_1(a) = t_2(a)$, then $t_1 = t_2$.

(32) If F_1 is naturally transformable to F_2 and F_2 is naturally transformable to F_3, then $G \cdot (s' \circ s) = G \cdot s' \circ G \cdot s$.

(33) If G_1 is naturally transformable to G_2 and G_2 is naturally transformable to G_3, then $(t' \circ t) \cdot F = t' \cdot F \circ t \cdot F$.

(34) If H_1 is naturally transformable to H_2, then $(u \cdot G) \cdot F = u \cdot (G \cdot F)$.

(35) If G_1 is naturally transformable to G_2, then $(H \cdot t) \cdot F = H \cdot (t \cdot F)$.

(36) If F_1 is naturally transformable to F_2, then $(H \cdot G) \cdot s = H \cdot (G \cdot s)$.

(37) $\mathrm{id}_G \cdot F = \mathrm{id}_{G \cdot F}$.

(38) $G \cdot \mathrm{id}_F = \mathrm{id}_{G \cdot F}$.

(39) If G_1 is naturally transformable to G_2, then $t \cdot \mathrm{id}_B = t$.

(40) If F_1 is naturally transformable to F_2, then $\mathrm{id}_B \cdot s = s$.

Let us consider A, B, C, F_1, F_2, G_1, G_2 and let us consider s, t. The functor $t\, s$ yields a natural transformation from $G_1 \cdot F_1$ to $G_2 \cdot F_2$ and is defined by:

(Def. 9) $t\, s = t \cdot F_2 \circ G_1 \cdot s$.

The following propositions are true:

(41) If F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2, then $t\, s = G_2 \cdot s \circ t \cdot F_1$.

(42) If F_1 is naturally transformable to F_2, then $\mathrm{id}_{\mathrm{id}_B}\, s = s$.

(43) If G_1 is naturally transformable to G_2, then $t\, \mathrm{id}_{\mathrm{id}_B} = t$.

(44) Suppose F_1 is naturally transformable to F_2 and G_1 is naturally transformable to G_2 and H_1 is naturally transformable to H_2. Then $u\,(t\, s) = (u\, t)\, s$.

(45) If G_1 is naturally transformable to G_2, then $t \cdot F = t\, \mathrm{id}_F$.

(46) If F_1 is naturally transformable to F_2, then $G \cdot s = \mathrm{id}_G\, s$.

(47) Suppose that

(i) F_1 is naturally transformable to F_2,

(ii) F_2 is naturally transformable to F_3,

(iii) G_1 is naturally transformable to G_2, and

(iv) G_2 is naturally transformable to G_3.

Then $(t' \circ t)\,(s' \circ s) = t'\, s' \circ t\, s$.

(48) Let F be a functor from A to B, G be a functor from C to D, and I, J be functors from B to C. If $I \cong J$, then $G \cdot I \cong G \cdot J$ and $I \cdot F \cong J \cdot F$.

(49) Let F be a functor from A to B, G be a functor from B to A, and I be a functor from A to A. If $I \cong \mathrm{id}_A$, then $F \cdot I \cong F$ and $I \cdot G \cong G$.

Let A, B be categories. We say that A is equivalent with B if and only if:

(Def. 10) There exists a functor F from A to B and there exists a functor G from B to A such that $G \cdot F \cong \mathrm{id}_A$ and $F \cdot G \cong \mathrm{id}_B$.

Let us notice that the predicate A is equivalent with B is reflexive and symmetric. We introduce A and B are equivalent as a synonym of A is equivalent with B.

Next we state two propositions:

(50) If $A \cong B$, then A is equivalent with B.

(53)[3] If A and B are equivalent and B and C are equivalent, then A and C are equivalent.

Let us consider A, B. Let us assume that A and B are equivalent. A functor from A to B is said to be an equivalence of A and B if:

(Def. 11) There exists a functor G from B to A such that $G \cdot \mathrm{it} \cong \mathrm{id}_A$ and $\mathrm{it} \cdot G \cong \mathrm{id}_B$.

Next we state several propositions:

(54) id_A is an equivalence of A and A.

(55) Suppose A and B are equivalent and B and C are equivalent. Let F be an equivalence of A and B and G be an equivalence of B and C. Then $G \cdot F$ is an equivalence of A and C.

(56) Suppose A and B are equivalent. Let F be an equivalence of A and B. Then there exists an equivalence G of B and A such that $G \cdot F \cong \mathrm{id}_A$ and $F \cdot G \cong \mathrm{id}_B$.

(57) For every functor F from A to B and for every functor G from B to A such that $G \cdot F \cong \mathrm{id}_A$ holds F is faithful.

(58) Suppose A and B are equivalent. Let F be an equivalence of A and B. Then
(i) F is full and faithful, and
(ii) for every object b of B there exists an object a of A such that b and $F(a)$ are isomorphic.

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[3] The propositions (51) and (52) have been removed.