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Vertex Sequences Induced by Chains[1]

Yatsuka Nakamura
Shinshu University
Department of Information Engineering
Nagano, Japan

Piotr Rudnicki
University of Alberta
Department of Computing Science
Edmonton, Alberta, Canada

Summary. In the three preliminary sections to the article we define two operations on finite sequences which seem to be of general interest. The first is the *cut* operation that extracts a contiguous chunk of a finite sequence from a position to a position. The second operation is a glueing catenation that given two finite sequences catenates them with removal of the first element of the second sequence. The main topic of the article is to define an operation which for a given chain in a graph returns the sequence of vertices through which the chain passes. We define the exact conditions when such an operation is uniquely definable. This is done with the help of the so called two-valued alternating finite sequences. We also prove theorems about the existence of simple chains which are subchains of a given chain. In order to do this we define the notion of a finite subsequence of a typed finite sequence.

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The articles [10], [13], [11], [14], [4], [5], [3], [6], [12], [1], [7], [2], [8], and [9] provide the notation and terminology for this paper.

1. PRELIMINARIES

We use the following convention: p, q denote finite sequences, X, Y denote sets, and i, k, l, m, n, r denote natural numbers.

The scheme *FinSegRng* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(i) : \mathcal{A} \leq i \land i \leq \mathcal{B} \land \mathcal{P}[i]\}$ is finite

for all values of the parameters.

One can prove the following propositions:

(1) $m+1 \leq k$ and $k \leq n$ iff there exists a natural number i such that $m \leq i$ and $i < n$ and $k = i+1$.

(2) If $q = p \upharpoonright \mathrm{Seg}\, n$, then $\mathrm{len}\, q \leq \mathrm{len}\, p$ and for every i such that $1 \leq i$ and $i \leq \mathrm{len}\, q$ holds $p(i) = q(i)$.

(3) If $X \subseteq \mathrm{Seg}\, k$ and $Y \subseteq \mathrm{dom}\, \mathrm{Sgm}\, X$, then $\mathrm{Sgm}\, X \cdot \mathrm{Sgm}\, Y = \mathrm{Sgm}\, \mathrm{rng}(\mathrm{Sgm}\, X \upharpoonright Y)$.

(4) For all natural numbers m, n holds $\overline{\overline{\{k : m \leq k \land k \leq m+n\}}} = n+1$.

(5) For every l such that $1 \leq l$ and $l \leq n$ holds $(\mathrm{Sgm}\{k_1; k_1$ ranges over natural numbers: $m+1 \leq k_1 \land k_1 \leq m+n\})(l) = m+l$.

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2. THE CUT OPERATION FOR FINITE SEQUENCES

Let p be a finite sequence and let m, n be natural numbers. The functor $\langle p(m),\ldots,p(n)\rangle$ yields a finite sequence and is defined by:

(Def. 1)(i) $\mathrm{len}\langle p(m),\ldots,p(n)\rangle + m = n+1$ and for every natural number i such that $i < \mathrm{len}\langle p(m),\ldots,p(n)\rangle$ holds $\langle p(m),\ldots,p(n)\rangle(i+1) = p(m+i)$ if $1 \leq m$ and $m \leq n+1$ and $n \leq \mathrm{len}\, p$,

(ii) $\langle p(m),\ldots,p(n)\rangle = \emptyset$, otherwise.

We now state several propositions:

(6) If $1 \leq m$ and $m \leq \mathrm{len}\, p$, then $\langle p(m),\ldots,p(m)\rangle = \langle p(m)\rangle$.

(7) $\langle p(1),\ldots,p(\mathrm{len}\, p)\rangle = p$.

(8) If $m \leq n$ and $n \leq r$ and $r \leq \mathrm{len}\, p$, then $\langle p(m+1),\ldots,p(n)\rangle ^\frown \langle p(n+1),\ldots,p(r)\rangle = \langle p(m+1),\ldots,p(r)\rangle$.

(9) If $m \leq \mathrm{len}\, p$, then $\langle p(1),\ldots,p(m)\rangle ^\frown \langle p(m+1),\ldots,p(\mathrm{len}\, p)\rangle = p$.

(10) If $m \leq n$ and $n \leq \mathrm{len}\, p$, then $\langle p(1),\ldots,p(m)\rangle ^\frown \langle p(m+1),\ldots,p(n)\rangle ^\frown \langle p(n+1),\ldots,p(\mathrm{len}\, p)\rangle = p$.

(11) $\mathrm{rng}\langle p(m),\ldots,p(n)\rangle \subseteq \mathrm{rng}\, p$.

Let D be a set, let p be a finite sequence of elements of D, and let m, n be natural numbers. Then $\langle p(m),\ldots,p(n)\rangle$ is a finite sequence of elements of D.

We now state the proposition

(12) If $1 \leq m$ and $m \leq n$ and $n \leq \mathrm{len}\, p$, then $\langle p(m),\ldots,p(n)\rangle(1) = p(m)$ and $\langle p(m),\ldots,p(n)\rangle(\mathrm{len}\langle p(m),\ldots,p(n)\rangle) = p(n)$.

3. THE GLUEING CATENATION OF FINITE SEQUENCES

Let p, q be finite sequences. The functor $p \frown\frown q$ yields a finite sequence and is defined as follows:

(Def. 2) $p \frown\frown q = p ^\frown \langle q(2),\ldots,q(\mathrm{len}\, q)\rangle$.

Next we state several propositions:

(13) If $q \neq \emptyset$, then $\mathrm{len}(p \frown\frown q) + 1 = \mathrm{len}\, p + \mathrm{len}\, q$.

(14) If $1 \leq k$ and $k \leq \mathrm{len}\, p$, then $(p \frown\frown q)(k) = p(k)$.

(15) If $1 \leq k$ and $k < \mathrm{len}\, q$, then $(p \frown\frown q)(\mathrm{len}\, p + k) = q(k+1)$.

(16) If $1 < \mathrm{len}\, q$, then $(p \frown\frown q)(\mathrm{len}(p \frown\frown q)) = q(\mathrm{len}\, q)$.

(17) $\mathrm{rng}(p \frown\frown q) \subseteq \mathrm{rng}\, p \cup \mathrm{rng}\, q$.

Let D be a set and let p, q be finite sequences of elements of D. Then $p \frown\frown q$ is a finite sequence of elements of D.

We now state the proposition

(18) If $p \neq \emptyset$ and $q \neq \emptyset$ and $p(\mathrm{len}\, p) = q(1)$, then $\mathrm{rng}(p \frown\frown q) = \mathrm{rng}\, p \cup \mathrm{rng}\, q$.

4. TWO VALUED ALTERNATING FINITE SEQUENCES

Let f be a finite sequence. We say that f is two-valued if and only if:

(Def. 3) $\operatorname{card}\operatorname{rng} f = 2$.

Next we state the proposition

(19) p is two-valued iff $\operatorname{len} p > 1$ and there exist sets x, y such that $x \neq y$ and $\operatorname{rng} p = \{x, y\}$.

Let f be a finite sequence. We say that f is alternating if and only if:

(Def. 4) For every natural number i such that $1 \leq i$ and $i+1 \leq \operatorname{len} f$ holds $f(i) \neq f(i+1)$.

Let us note that there exists a finite sequence which is two-valued and alternating.
In the sequel a, a_1, a_2 denote two-valued alternating finite sequences.
We now state four propositions:

(20) If $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$ and $a_1(1) = a_2(1)$, then $a_1 = a_2$.

(21) If $a_1 \neq a_2$ and $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$, then for every i such that $1 \leq i$ and $i \leq \operatorname{len} a_1$ holds $a_1(i) \neq a_2(i)$.

(22) If $a_1 \neq a_2$ and $\operatorname{len} a_1 = \operatorname{len} a_2$ and $\operatorname{rng} a_1 = \operatorname{rng} a_2$, then for every a such that $\operatorname{len} a = \operatorname{len} a_1$ and $\operatorname{rng} a = \operatorname{rng} a_1$ holds $a = a_1$ or $a = a_2$.

(23) If $X \neq Y$ and $n > 1$, then there exists a_1 such that $\operatorname{rng} a_1 = \{X, Y\}$ and $\operatorname{len} a_1 = n$ and $a_1(1) = X$.

5. FINITE SUBSEQUENCE OF FINITE SEQUENCES

Let us consider X and let f_1 be a finite sequence of elements of X. A finite subsequence is said to be a FinSubsequence of f_1 if:

(Def. 5) It $\subseteq f_1$.

In the sequel s_1 denotes a finite subsequence.
The following two propositions are true:

(24) If s_1 is a finite sequence, then $\operatorname{Seq} s_1 = s_1$.

(26)[1] Let f be a finite subsequence and g, h, f_2, f_3, f_4 be finite sequences. If $\operatorname{rng} g \subseteq \operatorname{dom} f$ and $\operatorname{rng} h \subseteq \operatorname{dom} f$ and $f_2 = f \cdot g$ and $f_3 = f \cdot h$ and $f_4 = f \cdot (g \frown h)$, then $f_4 = f_2 \frown f_3$.

We adopt the following rules: f_1, f_5, f_6 denote finite sequences of elements of X and f_7, f_8 denote FinSubsequences of f_1.
One can prove the following four propositions:

(27) $\operatorname{dom} f_7 \subseteq \operatorname{dom} f_1$ and $\operatorname{rng} f_7 \subseteq \operatorname{rng} f_1$.

(28) f_1 is a FinSubsequence of f_1.

(29) $f_7 \restriction Y$ is a FinSubsequence of f_1.

(30) For every FinSubsequence f_9 of f_5 such that $\operatorname{Seq} f_7 = f_5$ and $\operatorname{Seq} f_9 = f_6$ and $f_8 = f_7 \restriction \operatorname{rng}(\operatorname{Sgm} \operatorname{dom} f_7 \restriction \operatorname{dom} f_9)$ holds $\operatorname{Seq} f_8 = f_6$.

[1] The proposition (25) has been removed.

6. VERTEX SEQUENCES INDUCED BY CHAINS

In the sequel G is a graph.

Let us consider G. One can check that the vertices of G is non empty.

We adopt the following rules: v, v_1, v_2, v_3, v_4 are elements of the vertices of G and e is a set.

One can prove the following two propositions:

(31) If e joins v_1 with v_2, then e joins v_2 with v_1.

(32) If e joins v_1 with v_2 and e joins v_3 with v_4, then $v_1 = v_3$ and $v_2 = v_4$ or $v_1 = v_4$ and $v_2 = v_3$.

We follow the rules: v_5, v_6, v_7 denote finite sequences of elements of the vertices of G and c, c_1, c_2 denote chains of G.

Let us consider G. Note that there exists a chain of G which is empty.

Let us consider G, X. The functor G-VSet(X) yields a set and is defined by:

(Def. 6) $G\text{-VSet}(X) = \{v : \bigvee_{e:\text{element of the edges of } G} (e \in X \wedge (v = (\text{the source of } G)(e) \vee v = (\text{the target of } G)(e)))\}$.

Let us consider G, v_5 and let c be a finite sequence. We say that v_5 is vertex sequence of c if and only if:

(Def. 7) $\operatorname{len} v_5 = \operatorname{len} c + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} c$ holds $c(n)$ joins $(v_5)_n$ with $(v_5)_{n+1}$.

We now state four propositions:

(34)[2] If $c \neq \emptyset$ and v_5 is vertex sequence of c, then $G\text{-VSet}(\operatorname{rng} c) = \operatorname{rng} v_5$.

(35) $\langle v \rangle$ is vertex sequence of $\emptyset$.

(36) There exists v_5 which is vertex sequence of c.

(37) Suppose $c \neq \emptyset$ and v_6 is vertex sequence of c and v_7 is vertex sequence of c and $v_6 \neq v_7$. Then $v_6(1) \neq v_7(1)$ and for every v_5 such that v_5 is vertex sequence of c holds $v_5 = v_6$ or $v_5 = v_7$.

Let us consider G and let c be a finite sequence. We say that c alternates vertices in G if and only if:

(Def. 8) $\operatorname{len} c \geq 1$ and $\overline{\overline{G\text{-VSet}(\operatorname{rng} c)}} = 2$ and for every n such that $n \in \operatorname{dom} c$ holds (the source of $G)(c(n)) \neq$ (the target of $G)(c(n))$.

Next we state several propositions:

(38) If c alternates vertices in G and v_5 is vertex sequence of c, then for every k such that $k \in \operatorname{dom} c$ holds $v_5(k) \neq v_5(k+1)$.

(39) Suppose c alternates vertices in G and v_5 is vertex sequence of c. Then $\operatorname{rng} v_5 = \{(\text{the source of } G)(c(1)), (\text{the target of } G)(c(1))\}$.

(40) Suppose c alternates vertices in G and v_5 is vertex sequence of c. Then v_5 is a two-valued alternating finite sequence.

(41) Suppose c alternates vertices in G. Then there exist v_6, v_7 such that

(i) $v_6 \neq v_7$,

(ii) v_6 is vertex sequence of c,

(iii) v_7 is vertex sequence of c, and

(iv) for every v_5 such that v_5 is vertex sequence of c holds $v_5 = v_6$ or $v_5 = v_7$.

[2] The proposition (33) has been removed.

(42) Suppose v_5 is vertex sequence of c. Then $\overline{\overline{\text{the vertices of } G}} = 1$ or $c \neq \emptyset$ and c does not alternate vertices in G if and only if for every v_6 such that v_6 is vertex sequence of c holds $v_6 = v_5$.

Let us consider G, c. Let us assume that $\overline{\overline{\text{the vertices of } G}} = 1$ or $c \neq \emptyset$ and c does not alternate vertices in G. The functor vertex-seq(c) yielding a finite sequence of elements of the vertices of G is defined as follows:

(Def. 9) vertex-seq(c) is vertex sequence of c.

Next we state several propositions:

(43) If v_5 is vertex sequence of c and $c_1 = c \restriction \operatorname{Seg} n$ and $v_6 = v_5 \restriction \operatorname{Seg}(n+1)$, then v_6 is vertex sequence of c_1.

(44) If $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $q = \langle c(m), \ldots, c(n) \rangle$, then q is a chain of G.

(45) If $1 \leq m$ and $m \leq n$ and $n \leq \operatorname{len} c$ and $c_1 = \langle c(m), \ldots, c(n) \rangle$ and v_5 is vertex sequence of c and $v_6 = \langle v_5(m), \ldots, v_5(n+1) \rangle$, then v_6 is vertex sequence of c_1.

(46) If v_6 is vertex sequence of c_1 and v_7 is vertex sequence of c_2 and $v_6(\operatorname{len} v_6) = v_7(1)$, then $c_1 \frown c_2$ is a chain of G.

(47) Suppose v_6 is vertex sequence of c_1 and v_7 is vertex sequence of c_2 and $v_6(\operatorname{len} v_6) = v_7(1)$ and $c = c_1 \frown c_2$ and $v_5 = v_6 \frown\!\frown v_7$. Then v_5 is vertex sequence of c.

7. Vertex sequences induced by simple chains, paths and ordered chains

Let us consider G and let I_1 be a chain of G. We say that I_1 is simple if and only if:

(Def. 10) There exists v_5 such that v_5 is vertex sequence of I_1 and for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \operatorname{len} v_5$.

Let us consider G. One can verify that there exists a chain of G which is simple.

In the sequel s_2 denotes a simple chain of G.

One can prove the following propositions:

(49)[3] $s_2 \restriction \operatorname{Seg} n$ is a simple chain of G.

(50) If $2 < \operatorname{len} s_2$ and v_6 is vertex sequence of s_2 and v_7 is vertex sequence of s_2, then $v_6 = v_7$.

(51) If v_5 is vertex sequence of s_2, then for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} v_5$ and $v_5(n) = v_5(m)$ holds $n = 1$ and $m = \operatorname{len} v_5$.

(52) Suppose c is not a simple chain of G and v_5 is vertex sequence of c. Then there exists a FinSubsequence f_{10} of c and there exists a FinSubsequence f_{11} of v_5 and there exist c_1, v_6 such that $\operatorname{len} c_1 < \operatorname{len} c$ and v_6 is vertex sequence of c_1 and $\operatorname{len} v_6 < \operatorname{len} v_5$ and $v_5(1) = v_6(1)$ and $v_5(\operatorname{len} v_5) = v_6(\operatorname{len} v_6)$ and $\operatorname{Seq} f_{10} = c_1$ and $\operatorname{Seq} f_{11} = v_6$.

(53) Suppose v_5 is vertex sequence of c. Then there exists a FinSubsequence f_{10} of c and there exists a FinSubsequence f_{11} of v_5 and there exist s_2, v_6 such that $\operatorname{Seq} f_{10} = s_2$ and $\operatorname{Seq} f_{11} = v_6$ and v_6 is vertex sequence of s_2 and $v_5(1) = v_6(1)$ and $v_5(\operatorname{len} v_5) = v_6(\operatorname{len} v_6)$.

Let us consider G. One can verify that every chain of G which is empty is also one-to-one.

Next we state the proposition

(54) If p is a path of G, then $p \restriction \operatorname{Seg} n$ is a path of G.

Let us consider G. Observe that there exists a path of G which is simple.

One can prove the following two propositions:

[3] The proposition (48) has been removed.

(55) If $2 < \operatorname{len} s_2$, then s_2 is a path of G.

(56) s_2 is a path of G iff $\operatorname{len} s_2 = 0$ or $\operatorname{len} s_2 = 1$ or $s_2(1) \neq s_2(2)$.

Let us consider G. Observe that every chain of G which is empty is also oriented.

Let us consider G and let o_1 be an oriented chain of G. Let us assume that $o_1 \neq \emptyset$. The functor vertex-seq(o_1) yielding a finite sequence of elements of the vertices of G is defined as follows:

(Def. 11) vertex-seq(o_1) is vertex sequence of o_1 and $(\text{vertex-seq}(o_1))(1) = (\text{the source of } G)(o_1(1))$.

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