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Finite Topological Spaces

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Summary. By borrowing the concept of neighbourhood from the theory of topological space in continuous cases and extending it to a discrete case such as a space of lattice points we have defined such concepts as boundaries, closures, interiors, isolated points, and connected points as in the case of continuity. We have proved various properties which are satisfied by these concepts.

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The articles [12], [7], [16], [3], [13], [9], [2], [10], [18], [17], [5], [6], [8], [11], [14], [4], [15], and [1] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) Let A be a set and f be a finite sequence of elements of 2^A. Suppose that for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f_i \subseteq f_{i+1}$. Let i, j be natural numbers. If $i \leq j$ and $1 \leq i$ and $j \leq \operatorname{len} f$, then $f_i \subseteq f_j$.

(2) Let A be a set and f be a finite sequence of elements of 2^A. Suppose that for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $f_i \subseteq f_{i+1}$. Let i, j be natural numbers. Suppose $i < j$ and $1 \leq i$ and $j \leq \operatorname{len} f$ and $f_j \subseteq f_i$. Let k be a natural number. If $i \leq k$ and $k \leq j$, then $f_j = f_k$.

(3) Let F be a set. Suppose F is finite and $F \neq \emptyset$ and F is $\subseteq$-linear. Then there exists a set m such that $m \in F$ and for every set C such that $C \in F$ holds $C \subseteq m$.

(5)[1] Let f be a function. Suppose that for every natural number i holds $f(i) \subseteq f(i+1)$. Let i, j be natural numbers. If $i \leq j$, then $f(i) \subseteq f(j)$.

The scheme *MaxFinSeqEx* deals with a non empty set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, a subset $\mathcal{C}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$, and states that:

There exists a finite sequence f of elements of $2^{\mathcal{A}}$ such that
- (i) $\operatorname{len} f > 0$,
- (ii) $f_1 = \mathcal{C}$,
- (iii) for every natural number i such that $i > 0$ and $i < \operatorname{len} f$ holds $f_{i+1} = \mathcal{F}(f_i)$,
- (iv) $\mathcal{F}(f_{\operatorname{len} f}) = f_{\operatorname{len} f}$, and
- (v) for all natural numbers i, j such that $i > 0$ and $i < j$ and $j \leq \operatorname{len} f$ holds $f_i \subseteq \mathcal{B}$ and $f_i \subset f_j$

provided the following conditions are satisfied:
- $\mathcal{B}$ is finite,

[1] The proposition (4) has been removed.

- $\mathcal{C} \subseteq \mathcal{B}$, and
- For every subset A of $\mathcal{A}$ such that $A \subseteq \mathcal{B}$ holds $A \subseteq \mathcal{F}(A)$ and $\mathcal{F}(A) \subseteq \mathcal{B}$.

We consider finite topology spaces as extensions of 1-sorted structure as systems
$\langle$ a carrier, a neighbour-map $\rangle$,
where the carrier is a set and the neighbour-map is a function from the carrier into $2^{\text{the carrier}}$.

Let us note that there exists a finite topology space which is non empty and strict.

In the sequel F_1 denotes a non empty finite topology space and x, y denote elements of F_1.

Let F_1 be a non empty finite topology space and let x be an element of F_1. The functor $U(x)$ yields a subset of F_1 and is defined by:

(Def. 1) $U(x) = (\text{the neighbour-map of } F_1)(x)$.

Let x be a set and let y be a subset of $\{x\}$. Then $x \mapsto y$ is a function from $\{x\}$ into $2^{\{x\}}$.

The strict finite topology space $\mathrm{FT}_{\{0\}}$ is defined by:

(Def. 2) $\mathrm{FT}_{\{0\}} = \langle \{0\}, 0 \mapsto \Omega_{\{0\}} \rangle$.

Let us mention that $\mathrm{FT}_{\{0\}}$ is non empty.

Let I_1 be a non empty finite topology space. We say that I_1 is filled if and only if:

(Def. 3) For every element x of I_1 holds $x \in U(x)$.

One can prove the following two propositions:

(7)[2] $\mathrm{FT}_{\{0\}}$ is filled.

(8) $\mathrm{FT}_{\{0\}}$ is finite.

Let us observe that there exists a non empty finite topology space which is finite, filled, and strict.

Let T be a 1-sorted structure and let F be a set. We say that F is a cover of T if and only if:

(Def. 5)[3] The carrier of $T \subseteq \bigcup F$.

Next we state the proposition

(9) For every filled non empty finite topology space F_1 holds $\{U(x) : x \text{ ranges over elements of } F_1\}$ is a cover of F_1.

In the sequel A is a subset of F_1.

Let us consider F_1 and let A be a subset of F_1. The functor A^{δ} yields a subset of F_1 and is defined as follows:

(Def. 6) $A^{\delta} = \{x : U(x) \text{ meets } A \wedge U(x) \text{ meets } A^{\mathrm{c}}\}$.

Next we state the proposition

(10) $x \in A^{\delta}$ iff $U(x)$ meets A and $U(x)$ meets A^{c}.

Let us consider F_1 and let A be a subset of F_1. The functor A^{δ_i} yields a subset of F_1 and is defined as follows:

(Def. 7) $A^{\delta_i} = A \cap A^{\delta}$.

The functor A^{δ_o} yielding a subset of F_1 is defined by:

(Def. 8) $A^{\delta_o} = A^{\mathrm{c}} \cap A^{\delta}$.

The following proposition is true

(11) $A^{\delta} = A^{\delta_i} \cup A^{\delta_o}$.

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[2] The proposition (6) has been removed.
[3] The definition (Def. 4) has been removed.

Let us consider F_1 and let A be a subset of F_1. The functor A^i yielding a subset of F_1 is defined as follows:

(Def. 9) $A^i = \{x : U(x) \subseteq A\}$.

The functor A^b yielding a subset of F_1 is defined by:

(Def. 10) $A^b = \{x : U(x) \text{ meets } A\}$.

The functor A^s yielding a subset of F_1 is defined by:

(Def. 11) $A^s = \{x : x \in A \ \wedge\ U(x) \setminus \{x\} \text{ misses } A\}$.

Let us consider F_1 and let A be a subset of F_1. The functor A^n yields a subset of F_1 and is defined as follows:

(Def. 12) $A^n = A \setminus A^s$.

The functor A^f yielding a subset of F_1 is defined by:

(Def. 13) $A^f = \{x : \bigvee_y \ (y \in A \ \wedge\ x \in U(y))\}$.

Let I_1 be a non empty finite topology space. We say that I_1 is symmetric if and only if:

(Def. 14) For all elements x, y of I_1 such that $y \in U(x)$ holds $x \in U(y)$.

The following propositions are true:

(12) $x \in A^i$ iff $U(x) \subseteq A$.

(13) $x \in A^b$ iff $U(x)$ meets A.

(14) $x \in A^s$ iff $x \in A$ and $U(x) \setminus \{x\}$ misses A.

(15) $x \in A^n$ iff $x \in A$ and $U(x) \setminus \{x\}$ meets A.

(16) $x \in A^f$ iff there exists y such that $y \in A$ and $x \in U(y)$.

(17) F_1 is symmetric iff for every A holds $A^b = A^f$.

In the sequel F is a subset of F_1.

Let us consider F_1 and let I_1 be a subset of F_1. We say that I_1 is open if and only if:

(Def. 15) $I_1 = {I_1}^i$.

We say that I_1 is closed if and only if:

(Def. 16) $I_1 = {I_1}^b$.

We say that I_1 is connected if and only if:

(Def. 17) For all subsets B, C of F_1 such that $I_1 = B \cup C$ and $B \neq \emptyset$ and $C \neq \emptyset$ and B misses C holds B^b meets C.

Let us consider F_1 and let A be a subset of F_1. The functor A^{f_b} yields a subset of F_1 and is defined by:

(Def. 18) $A^{f_b} = \bigcap\{F : A \subseteq F \ \wedge\ F \text{is closed}\}$.

The functor A^{f_i} yields a subset of F_1 and is defined as follows:

(Def. 19) $A^{f_i} = \bigcup\{F : A \subseteq F \ \wedge\ F \text{is open}\}$.

Next we state a number of propositions:

(18) For every filled non empty finite topology space F_1 and for every subset A of F_1 holds $A \subseteq A^b$.

(19) For every non empty finite topology space F_1 and for all subsets A, B of F_1 such that $A \subseteq B$ holds $A^b \subseteq B^b$.

(20) Let F_1 be a filled finite non empty finite topology space and A be a subset of F_1. Then A is connected if and only if for every element x of F_1 such that $x \in A$ there exists a finite sequence S of elements of $2^{\text{the carrier of } F_1}$ such that $\operatorname{len} S > 0$ and $S_1 = \{x\}$ and for every natural number i such that $i > 0$ and $i < \operatorname{len} S$ holds $S_{i+1} = (S_i)^b \cap A$ and $A \subseteq S_{\operatorname{len} S}$.

(21) For every non empty set E and for every subset A of E and for every element x of E holds $x \in A^{\mathrm{c}}$ iff $x \notin A$.

(22) $((A^{\mathrm{c}})^i)^{\mathrm{c}} = A^b$.

(23) $((A^{\mathrm{c}})^b)^{\mathrm{c}} = A^i$.

(24) $A^\delta = A^b \cap (A^{\mathrm{c}})^b$.

(25) $(A^{\mathrm{c}})^\delta = A^\delta$.

(26) If $x \in A^s$, then $x \notin (A \setminus \{x\})^b$.

(27) If $A^s \neq \emptyset$ and $\overline{\overline{A}} \neq 1$, then A is not connected.

(28) For every filled non empty finite topology space F_1 and for every subset A of F_1 holds $A^i \subseteq A$.

(29) For every set E and for all subsets A, B of E holds $A = B$ iff $A^{\mathrm{c}} = B^{\mathrm{c}}$.

(30) If A is open, then A^{c} is closed.

(31) If A is closed, then A^{c} is open.

REFERENCES

[1] Grzegorz Bancerek. Cardinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/card_1.html`.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[3] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[8] Czesław Byliński. A classical first order language. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/cqc_lang.html`.

[9] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[10] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[11] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[12] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[13] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[14] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_1.html`.

[15] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_4.html.

[16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[17] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[18] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

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