

The Euclidean Space

Agata Darmochwał
Warsaw University
Białystok

Summary. The general definition of Euclidean Space.

MML Identifier: EUCLID.

WWW: <http://mizar.org/JFM/Vol3/euclid.html>

The articles [6], [16], [1], [14], [4], [11], [5], [10], [13], [2], [8], [7], [15], [9], [12], and [3] provide the notation and terminology for this paper.

In this paper k, n denote natural numbers and r denotes a real number.

Let us consider n . The functor \mathcal{R}^n yields a non empty set of finite sequences of \mathbb{R} and is defined as follows:

(Def. 1) $\mathcal{R}^n = \mathbb{R}^n$.

In the sequel x denotes a finite sequence of elements of \mathbb{R} .

The function $|\square|_{\mathbb{R}}$ from \mathbb{R} into \mathbb{R} is defined by:

(Def. 2) For every r holds $|\square|_{\mathbb{R}}(r) = |r|$.

Let us consider x . The functor $|x|$ yielding a finite sequence of elements of \mathbb{R} is defined as follows:

(Def. 3) $|x| = |\square|_{\mathbb{R}} \cdot x$.

Let us consider n . The functor $\underbrace{\langle 0, \dots, 0 \rangle}_n$ yields a finite sequence of elements of \mathbb{R} and is defined as follows:

(Def. 4) $\underbrace{\langle 0, \dots, 0 \rangle}_n = n \mapsto (0 \text{ qua real number})$.

Let us consider n . Then $\underbrace{\langle 0, \dots, 0 \rangle}_n$ is an element of \mathcal{R}^n .

In the sequel x, x_1, x_2, y denote elements of \mathcal{R}^n .

Let us consider n, x . Then $-x$ is an element of \mathcal{R}^n .

Let us consider n, x, y . Then $x + y$ is an element of \mathcal{R}^n . Then $x - y$ is an element of \mathcal{R}^n .

Let us consider n , let r be a real number, and let us consider x . Then $r \cdot x$ is an element of \mathcal{R}^n .

Let us consider n, x . Then $|x|$ is an element of \mathbb{R}^n .

Let us consider n, x . Then 2x is an element of \mathbb{R}^n .

Let x be a finite sequence of elements of \mathbb{R} . The functor $|x|$ yields a real number and is defined as follows:

(Def. 5) $|x| = \sqrt{\sum^2 x}$.

One can prove the following propositions:

- (2)¹ $\text{len } x = n$.
- (3) $\text{dom } x = \text{Seg } n$.
- (4) $x(k) \in \mathbb{R}$.
- (5) If for every k such that $k \in \text{Seg } n$ holds $x_1(k) = x_2(k)$, then $x_1 = x_2$.
- (6) If $r = x(k)$, then $|x|(k) = |r|$.
- (7) $|\underbrace{\langle 0, \dots, 0 \rangle}_n| = n \mapsto (0 \text{ qua real number})$.
- (8) $|-x| = |x|$.
- (9) $|r \cdot x| = |r| \cdot |x|$.
- (10) $|\underbrace{\langle 0, \dots, 0 \rangle}_n| = 0$.
- (11) If $|x| = 0$, then $x = \underbrace{\langle 0, \dots, 0 \rangle}_n$.
- (12) $|x| \geq 0$.
- (13) $|-x| = |x|$.
- (14) $|r \cdot x| = |r| \cdot |x|$.
- (15) $|x_1 + x_2| \leq |x_1| + |x_2|$.
- (16) $|x_1 - x_2| \leq |x_1| + |x_2|$.
- (17) $|x_1| - |x_2| \leq |x_1 + x_2|$.
- (18) $|x_1| - |x_2| \leq |x_1 - x_2|$.
- (19) $|x_1 - x_2| = 0$ iff $x_1 = x_2$.
- (20) If $x_1 \neq x_2$, then $|x_1 - x_2| > 0$.
- (21) $|x_1 - x_2| = |x_2 - x_1|$.
- (22) $|x_1 - x_2| \leq |x_1 - x| + |x - x_2|$.

Let us consider n . The functor ρ^n yielding a function from $[\mathcal{R}^n, \mathcal{R}^n]$ into \mathbb{R} is defined by:

(Def. 6) For all elements x, y of \mathcal{R}^n holds $\rho^n(x, y) = |x - y|$.

We now state two propositions:

- (23) ${}^2(x - y) = {}^2(y - x)$.
- (24) ρ^n is a metric of \mathcal{R}^n .

Let us consider n . The functor \mathcal{E}^n yields a strict metric space and is defined as follows:

(Def. 7) $\mathcal{E}^n = \langle \mathcal{R}^n, \rho^n \rangle$.

Let us consider n . One can check that \mathcal{E}^n is non empty.

Let us consider n . The functor \mathcal{E}_T^n yields a strict topological space and is defined by:

¹ The proposition (1) has been removed.

(Def. 8) $\mathcal{E}_T^n = (\mathcal{E}^n)_{\text{top}}$.

Let us consider n . Observe that \mathcal{E}_T^n is non empty.

We use the following convention: p, p_1, p_2, p_3 are points of \mathcal{E}_T^n and x, x_1, x_2, y, y_1, y_2 are real numbers.

The following four propositions are true:

- (25) The carrier of $\mathcal{E}_T^n = \mathcal{R}^n$.
- (26) p is a function from $\text{Seg } n$ into \mathbb{R} .
- (27) p is a finite sequence of elements of \mathbb{R} .
- (28) For every finite sequence f such that $f = p$ holds $\text{len } f = n$.

Let us consider n . The functor $0_{\mathcal{E}_T^n}$ yielding a point of \mathcal{E}_T^n is defined by:

(Def. 9) $0_{\mathcal{E}_T^n} = \underbrace{\langle 0, \dots, 0 \rangle}_n$.

Let us consider n, p_1, p_2 . The functor $p_1 + p_2$ yielding a point of \mathcal{E}_T^n is defined by:

(Def. 10) For all elements p'_1, p'_2 of \mathcal{R}^n such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 + p_2 = p'_1 + p'_2$.

Let us observe that the functor $p_1 + p_2$ is commutative.

One can prove the following propositions:

- (29) For every element x of \mathcal{R}^n holds ${}^2|x| = {}^2x$.
- (30) $(p_1 + p_2) + p_3 = p_1 + (p_2 + p_3)$.
- (31) $0_{\mathcal{E}_T^n} + p = p$ and $p + 0_{\mathcal{E}_T^n} = p$.

Let us consider x, n, p . The functor $x \cdot p$ yielding a point of \mathcal{E}_T^n is defined as follows:

(Def. 11) For every element p' of \mathcal{R}^n such that $p' = p$ holds $x \cdot p = x \cdot p'$.

Next we state several propositions:

- (32) $x \cdot 0_{\mathcal{E}_T^n} = 0_{\mathcal{E}_T^n}$.
- (33) $1 \cdot p = p$ and $0 \cdot p = 0_{\mathcal{E}_T^n}$.
- (34) $(x \cdot y) \cdot p = x \cdot (y \cdot p)$.
- (35) If $x \cdot p = 0_{\mathcal{E}_T^n}$, then $x = 0$ or $p = 0_{\mathcal{E}_T^n}$.
- (36) $x \cdot (p_1 + p_2) = x \cdot p_1 + x \cdot p_2$.
- (37) $(x + y) \cdot p = x \cdot p + y \cdot p$.
- (38) If $x \cdot p_1 = x \cdot p_2$, then $x = 0$ or $p_1 = p_2$.

Let us consider n, p . The functor $-p$ yielding a point of \mathcal{E}_T^n is defined by:

(Def. 12) For every element p' of \mathcal{R}^n such that $p' = p$ holds $-p = -p'$.

One can prove the following propositions:

- (39) $--p = p$.
- (40) $p + -p = 0_{\mathcal{E}_T^n}$.
- (41) If $p_1 + p_2 = 0_{\mathcal{E}_T^n}$, then $p_1 = -p_2$ and $p_2 = -p_1$.
- (42) $-(p_1 + p_2) = -p_1 + -p_2$.

$$(43) \quad -p = (-1) \cdot p.$$

$$(44) \quad -x \cdot p = (-x) \cdot p \text{ and } -x \cdot p = x \cdot -p.$$

Let us consider n, p_1, p_2 . The functor $p_1 - p_2$ yields a point of \mathcal{E}_1^n and is defined as follows:

(Def. 13) For all elements p'_1, p'_2 of \mathcal{R}^n such that $p'_1 = p_1$ and $p'_2 = p_2$ holds $p_1 - p_2 = p'_1 - p'_2$.

The following propositions are true:

$$(45) \quad p_1 - p_2 = p_1 + -p_2.$$

$$(46) \quad p - p = 0_{\mathcal{E}_1^n}.$$

$$(47) \quad \text{If } p_1 - p_2 = 0_{\mathcal{E}_1^n}, \text{ then } p_1 = p_2.$$

$$(48) \quad -(p_1 - p_2) = p_2 - p_1 \text{ and } -(p_1 - p_2) = -p_1 + p_2.$$

$$(49) \quad p_1 + (p_2 - p_3) = (p_1 + p_2) - p_3.$$

$$(50) \quad p_1 - (p_2 + p_3) = p_1 - p_2 - p_3.$$

$$(51) \quad p_1 - (p_2 - p_3) = (p_1 - p_2) + p_3.$$

$$(52) \quad p = (p + p_1) - p_1 \text{ and } p = (p - p_1) + p_1.$$

$$(53) \quad x \cdot (p_1 - p_2) = x \cdot p_1 - x \cdot p_2.$$

$$(54) \quad (x - y) \cdot p = x \cdot p - y \cdot p.$$

In the sequel p, p_1, p_2 denote points of \mathcal{E}_1^2 .

The following proposition is true

$$(55) \quad \text{There exist real numbers } x, y \text{ such that } p = \langle x, y \rangle.$$

Let us consider p . The functor p_1 yielding a real number is defined as follows:

(Def. 14) For every finite sequence f such that $p = f$ holds $p_1 = f(1)$.

The functor p_2 yielding a real number is defined by:

(Def. 15) For every finite sequence f such that $p = f$ holds $p_2 = f(2)$.

Let x, y be real numbers. The functor $[x, y]$ yielding a point of \mathcal{E}_1^2 is defined as follows:

(Def. 16) $[x, y] = \langle x, y \rangle$.

The following propositions are true:

$$(56) \quad [x, y]_1 = x \text{ and } [x, y]_2 = y.$$

$$(57) \quad p = [p_1, p_2].$$

$$(58) \quad 0_{\mathcal{E}_1^2} = [0, 0].$$

$$(59) \quad p_1 + p_2 = [(p_1)_1 + (p_2)_1, (p_1)_2 + (p_2)_2].$$

$$(60) \quad [x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2].$$

$$(61) \quad x \cdot p = [x \cdot p_1, x \cdot p_2].$$

$$(62) \quad x \cdot [x_1, y_1] = [x \cdot x_1, x \cdot y_1].$$

$$(63) \quad -p = [-p_1, -p_2].$$

$$(64) \quad -[x_1, y_1] = [-x_1, -y_1].$$

$$(65) \quad p_1 - p_2 = [(p_1)_1 - (p_2)_1, (p_1)_2 - (p_2)_2].$$

$$(66) \quad [x_1, y_1] - [x_2, y_2] = [x_1 - x_2, y_1 - y_2].$$

REFERENCES

- [1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. <http://mizar.org/JFM/Voll/ordinall.html>.
- [2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/finseq_1.html.
- [3] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/pcomps_1.html.
- [4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/funct_1.html.
- [5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/funct_2.html.
- [6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/zfmisc_1.html.
- [7] Czesław Byliński. Binary operations applied to finite sequences. *Journal of Formalized Mathematics*, 2, 1990. <http://mizar.org/JFM/Vol2/finseqop.html>.
- [8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/finseq_2.html.
- [9] Czesław Byliński. The sum and product of finite sequences of real numbers. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/rvsum_1.html.
- [10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/real_1.html.
- [11] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/metric_1.html.
- [12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/pre_topc.html.
- [13] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. <http://mizar.org/JFM/Voll/absvalue.html>.
- [14] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. <http://mizar.org/JFM/Addenda/numbers.html>.
- [15] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers operations: min, max, square, and square root. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/square_1.html.
- [16] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Voll/subset_1.html.

Received November 21, 1991

Published January 2, 2004
