JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

# On Defining Functions on Trees[1]

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**Summary.** The continuation of the sequence of articles on trees (see [2], [3], [4], [5]) and on context-free grammars ([13]). We define the set of complete parse trees for a given context-free grammar. Next we define the scheme of induction for the set and the scheme of defining functions by induction on the set. For each symbol of a context-free grammar we define the terminal, the pretraversal, and the posttraversal languages. The introduced terminology is tested on the example of Peano naturals.

MML Identifier: DTCONSTR.

WWW: http://mizar.org/JFM/Vol5/dtconstr.html

The articles [17], [10], [21], [19], [1], [23], [22], [8], [9], [6], [12], [14], [18], [15], [16], [7], [20], [13], [2], [3], [4], [5], and [11] provide the notation and terminology for this paper.

## 1. Preliminaries

The following propositions are true:

(1) For every non empty set $D$ holds every finite sequence of elements of $\mathrm{FinTrees}(D)$ is a finite sequence of elements of $\mathrm{Trees}(D)$.

(2) For all sets $x$, $y$ and for every finite sequence $p$ of elements of $x$ such that $y \in \mathrm{dom}\, p$ holds $p(y) \in x$.

Let $X$ be a set. Observe that every element of $X^*$ is relation-like and function-like.

Let $X$ be a set. Observe that every element of $X^*$ is finite sequence-like.

Let $D$ be a non empty set and let $t$ be an element of $\mathrm{FinTrees}(D)$. Note that $\mathrm{dom}\, t$ is finite.

Let $D$ be a non empty set and let $T$ be a set of trees decorated with elements of $D$. Note that every finite sequence of elements of $T$ is decorated tree yielding.

Let $D$ be a non empty set, let $F$ be a non empty set of trees decorated with elements of $D$, and let $T_1$ be a non empty subset of $F$. We see that the element of $T_1$ is an element of $F$.

Let $p$ be a finite sequence. Let us assume that $p$ is decorated tree yielding. The roots of $p$ constitute a finite sequence defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{dom}(\text{the roots of } p) = \mathrm{dom}\, p$, and

(ii) for every natural number $i$ such that $i \in \mathrm{dom}\, p$ there exists a decorated tree $T$ such that $T = p(i)$ and $(\text{the roots of } p)(i) = T(\emptyset)$.

[1]This work was partially supported by NSERC Grant OGP9207 while the first author visited University of Alberta, May–June 1993.

Let $D$ be a non empty set, let $T$ be a set of trees decorated with elements of $D$, and let $p$ be a finite sequence of elements of $T$. Then the roots of $p$ is a finite sequence of elements of $D$.

We now state four propositions:

(3) The roots of $\emptyset = \emptyset$.

(4) For every decorated tree $T$ holds the roots of $\langle T \rangle = \langle T(\emptyset) \rangle$.

(5) Let $D$ be a non empty set, $F$ be a subset of $\mathrm{FinTrees}(D)$, and $p$ be a finite sequence of elements of $F$. Suppose $\mathrm{len}(\text{the roots of } p) = 1$. Then there exists an element $x$ of $\mathrm{FinTrees}(D)$ such that $p = \langle x \rangle$ and $x \in F$.

(6) For all decorated trees $T_2$, $T_3$ holds the roots of $\langle T_2, T_3 \rangle = \langle T_2(\emptyset), T_3(\emptyset) \rangle$.

Let $f$ be a function. The functor $\mathrm{pr1}(f)$ yields a function and is defined by:

(Def. 2) $\mathrm{dom}\,\mathrm{pr1}(f) = \mathrm{dom}\, f$ and for every set $x$ such that $x \in \mathrm{dom}\, f$ holds $\mathrm{pr1}(f)(x) = f(x)_{\mathbf{1}}$.

The functor $\mathrm{pr2}(f)$ yielding a function is defined as follows:

(Def. 3) $\mathrm{dom}\,\mathrm{pr2}(f) = \mathrm{dom}\, f$ and for every set $x$ such that $x \in \mathrm{dom}\, f$ holds $\mathrm{pr2}(f)(x) = f(x)_{\mathbf{2}}$.

Let $X$, $Y$ be sets and let $f$ be a finite sequence of elements of $[:X, Y:]$. Then $\mathrm{pr1}(f)$ is a finite sequence of elements of $X$. Then $\mathrm{pr2}(f)$ is a finite sequence of elements of $Y$.

Next we state the proposition

(7) $\mathrm{pr1}(\emptyset) = \emptyset$ and $\mathrm{pr2}(\emptyset) = \emptyset$.

The scheme *MonoSetSeq* deals with a function $\mathcal{A}$, a set $\mathcal{B}$, and a binary functor $\mathcal{F}$ yielding a set, and states that:

For all natural numbers $k$, $s$ holds $\mathcal{A}(k) \subseteq \mathcal{A}(k+s)$

provided the parameters satisfy the following condition:

- For every natural number $n$ holds $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \mathcal{F}(n, \mathcal{A}(n))$.

## 2. THE SET OF PARSE TREES

Let $A$ be a non empty set and let $R$ be a relation between $A$ and $A^*$. Note that $\langle A, R \rangle$ is non empty.

Now we present two schemes. The scheme *DTConstrStrEx* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a strict non empty tree construction structure $G$ such that

(i) the carrier of $G = \mathcal{A}$, and

(ii) for every symbol $x$ of $G$ and for every finite sequence $p$ of elements of the carrier of $G$ holds $x \Rightarrow p$ iff $\mathcal{P}[x, p]$

for all values of the parameters.

The scheme *DTConstrStrUniq* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

Let $G_1$, $G_2$ be strict non empty tree construction structures. Suppose that

(i) the carrier of $G_1 = \mathcal{A}$,

(ii) for every symbol $x$ of $G_1$ and for every finite sequence $p$ of elements of the carrier of $G_1$ holds $x \Rightarrow p$ iff $\mathcal{P}[x, p]$,

(iii) the carrier of $G_2 = \mathcal{A}$, and

(iv) for every symbol $x$ of $G_2$ and for every finite sequence $p$ of elements of the carrier of $G_2$ holds $x \Rightarrow p$ iff $\mathcal{P}[x, p]$.

Then $G_1 = G_2$

for all values of the parameters.

We now state the proposition

(8) For every non empty tree construction structure $G$ holds the terminals of $G$ misses the nonterminals of $G$.

Now we present four schemes. The scheme *DTCMin* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset $X$ of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$ :]) such that

(i) $X = \bigcup \mathcal{A}$,

(ii) for every symbol $d$ of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d) \rangle \in X$,

(iii) for every symbol $o$ of $\mathcal{B}$ and for every finite sequence $p$ of elements of $X$ such that $o \Rightarrow$ pr1(the roots of $p$) holds $\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p)) \rangle$-tree$(p) \in X$, and

(iv) for every subset $F$ of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$ :]) such that for every symbol $d$ of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $\langle d, \mathcal{F}(d) \rangle \in F$ and for every symbol $o$ of $\mathcal{B}$ and for every finite sequence $p$ of elements of $F$ such that $o \Rightarrow$ pr1(the roots of $p$) holds $\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p)) \rangle$-tree$(p) \in F$ holds $X \subseteq F$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d \rangle$; $t$ ranges over symbols of $\mathcal{B}$, $d$ ranges over elements of $\mathcal{C}$ : $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let $n$ be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p)) \rangle$-tree$(p)$; $o$ ranges over symbols of $\mathcal{B}$, $p$ ranges over elements of $\mathcal{A}(n)^*$: $\bigvee_{q:\text{finite sequence of elements of FinTrees([: the carrier of } \mathcal{B}, \mathcal{C} \text{ :])}} (p = q \wedge o \Rightarrow \text{pr1(the roots of } q))\}$.

The scheme *DTCSymbols* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

There exists a subset $X_1$ of FinTrees(the carrier of $\mathcal{B}$) such that

(i) $X_1 = \{t_{\mathbf{1}}$; $t$ ranges over elements of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$ :]): $t \in \bigcup \mathcal{A}\}$,

(ii) for every symbol $d$ of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in X_1$,

(iii) for every symbol $o$ of $\mathcal{B}$ and for every finite sequence $p$ of elements of $X_1$ such that $o \Rightarrow$ the roots of $p$ holds $o$-tree$(p) \in X_1$, and

(iv) for every subset $F$ of FinTrees(the carrier of $\mathcal{B}$) such that for every symbol $d$ of $\mathcal{B}$ such that $d \in$ the terminals of $\mathcal{B}$ holds the root tree of $d \in F$ and for every symbol $o$ of $\mathcal{B}$ and for every finite sequence $p$ of elements of $F$ such that $o \Rightarrow$ the roots of $p$ holds $o$-tree$(p) \in F$ holds $X_1 \subseteq F$

provided the parameters meet the following conditions:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d \rangle$; $t$ ranges over symbols of $\mathcal{B}$, $d$ ranges over elements of $\mathcal{C}$ : $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let $n$ be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p)) \rangle$-tree$(p)$; $o$ ranges over symbols of $\mathcal{B}$, $p$ ranges over elements of $\mathcal{A}(n)^*$: $\bigvee_{q:\text{finite sequence of elements of FinTrees([: the carrier of } \mathcal{B}, \mathcal{C} \text{ :])}} (p = q \wedge o \Rightarrow \text{pr1(the roots of } q))\}$.

The scheme *DTCHeight* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

Let $n$ be a natural number and $d_1$ be an element of FinTrees([: the carrier of $\mathcal{B}$, $\mathcal{C}$ :]). If $d_1 \in \bigcup \mathcal{A}$, then $d_1 \in \mathcal{A}(n)$ iff height dom $d_1 \leq n$

provided the parameters meet the following requirements:

- $\operatorname{dom} \mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{$the root tree of $\langle t, d \rangle$; $t$ ranges over symbols of $\mathcal{B}$, $d$ ranges over elements of $\mathcal{C}$ : $t \in$ the terminals of $\mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let $n$ be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \text{pr1(the roots of } p), \text{pr2(the roots of } p)) \rangle$-tree$(p)$; $o$ ranges over symbols of $\mathcal{B}$, $p$ ranges over elements

of $\mathcal{A}(n)^*$: $\bigvee_{q:\text{finite sequence of elements of FinTrees}([:\text{the carrier of } \mathcal{B}, \mathcal{C}:])} (p = q \wedge o \Rightarrow \text{pr1}(\text{the roots of } q))\}$.

The scheme *DTCUniq* deals with a function $\mathcal{A}$, a non empty tree construction structure $\mathcal{B}$, a non empty set $\mathcal{C}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{C}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{C}$, and states that:

Let $d_2$, $d_3$ be trees decorated with elements of $[:\text{the carrier of } \mathcal{B}, \mathcal{C}:]$. If $d_2 \in \bigcup \mathcal{A}$ and $d_3 \in \bigcup \mathcal{A}$ and $(d_2)_{\mathbf{1}} = (d_3)_{\mathbf{1}}$, then $d_2 = d_3$

provided the following conditions are satisfied:

- $\text{dom}\,\mathcal{A} = \mathbb{N}$,
- $\mathcal{A}(0) = \{\text{the root tree of } \langle t, d\rangle;\ t \text{ ranges over symbols of } \mathcal{B}, d \text{ ranges over elements of } \mathcal{C} : t \in \text{the terminals of } \mathcal{B} \wedge d = \mathcal{F}(t) \vee t \Rightarrow \emptyset \wedge d = \mathcal{G}(t, \emptyset, \emptyset)\}$, and
- Let $n$ be a natural number. Then $\mathcal{A}(n+1) = \mathcal{A}(n) \cup \{\langle o, \mathcal{G}(o, \text{pr1}(\text{the roots of } p), \text{pr2}(\text{the roots of } p))\rangle\text{-tree}(p);\ o \text{ ranges over symbols of } \mathcal{B}, p \text{ ranges over elements of } \mathcal{A}(n)^*: \bigvee_{q:\text{finite sequence of elements of FinTrees}([:\text{the carrier of } \mathcal{B}, \mathcal{C}:])} (p = q \wedge o \Rightarrow \text{pr1}(\text{the roots of } q))\}$.

Let $G$ be a non empty tree construction structure. The functor $\text{TS}(G)$ yields a subset of FinTrees(the carrier of $G$) and is defined by the conditions (Def. 4).

(Def. 4)(i) For every symbol $d$ of $G$ such that $d \in$ the terminals of $G$ holds the root tree of $d \in \text{TS}(G)$,

(ii) for every symbol $o$ of $G$ and for every finite sequence $p$ of elements of $\text{TS}(G)$ such that $o \Rightarrow$ the roots of $p$ holds $o\text{-tree}(p) \in \text{TS}(G)$, and

(iii) for every subset $F$ of FinTrees(the carrier of $G$) such that for every symbol $d$ of $G$ such that $d \in$ the terminals of $G$ holds the root tree of $d \in F$ and for every symbol $o$ of $G$ and for every finite sequence $p$ of elements of $F$ such that $o \Rightarrow$ the roots of $p$ holds $o\text{-tree}(p) \in F$ holds $\text{TS}(G) \subseteq F$.

Now we present three schemes. The scheme *DTConstrInd* deals with a non empty tree construction structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every tree $t$ decorated with elements of the carrier of $\mathcal{A}$ such that $t \in \text{TS}(\mathcal{A})$ holds $\mathcal{P}[t]$

provided the following conditions are satisfied:

- For every symbol $s$ of $\mathcal{A}$ such that $s \in$ the terminals of $\mathcal{A}$ holds $\mathcal{P}[\text{the root tree of } s]$, and
- Let $n_1$ be a symbol of $\mathcal{A}$ and $t_1$ be a finite sequence of elements of $\text{TS}(\mathcal{A})$. Suppose that
  (i) $n_1 \Rightarrow$ the roots of $t_1$, and
  (ii) for every tree $t$ decorated with elements of the carrier of $\mathcal{A}$ such that $t \in \text{rng}\, t_1$ holds $\mathcal{P}[t]$.
  Then $\mathcal{P}[n_1\text{-tree}(t_1)]$.

The scheme *DTConstrIndDef* deals with a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and states that:

There exists a function $f$ from $\text{TS}(\mathcal{A})$ into $\mathcal{B}$ such that
(i) for every symbol $t$ of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $f(\text{the root tree of } t) = \mathcal{F}(t)$, and
(ii) for every symbol $n_1$ of $\mathcal{A}$ and for every finite sequence $t_1$ of elements of $\text{TS}(\mathcal{A})$ such that $n_1 \Rightarrow$ the roots of $t_1$ holds $f(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, \text{the roots of } t_1, f \cdot t_1)$

for all values of the parameters.

The scheme *DTConstrUniqDef* deals with a non empty tree construction structure $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a ternary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and functions $\mathcal{C}$, $\mathcal{D}$ from $\text{TS}(\mathcal{A})$ into $\mathcal{B}$, and states that:

$\mathcal{C} = \mathcal{D}$

provided the following conditions are met:

- (i) For every symbol $t$ of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{C}(\text{the root tree of } t) = \mathcal{F}(t)$, and

(ii) for every symbol $n_1$ of $\mathcal{A}$ and for every finite sequence $t_1$ of elements of $\mathrm{TS}(\mathcal{A})$ such that $n_1 \Rightarrow$ the roots of $t_1$ holds $\mathcal{C}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, \text{the roots of } t_1, \mathcal{C} \cdot t_1)$, and

- (i) For every symbol $t$ of $\mathcal{A}$ such that $t \in$ the terminals of $\mathcal{A}$ holds $\mathcal{D}$(the root tree of $t) = \mathcal{F}(t)$, and
 (ii) for every symbol $n_1$ of $\mathcal{A}$ and for every finite sequence $t_1$ of elements of $\mathrm{TS}(\mathcal{A})$ such that $n_1 \Rightarrow$ the roots of $t_1$ holds $\mathcal{D}(n_1\text{-tree}(t_1)) = \mathcal{G}(n_1, \text{the roots of } t_1, \mathcal{D} \cdot t_1)$.

## 3. An example: Peano naturals

The strict non empty tree construction structure $\mathbb{N}_{\mathrm{Peano}}$ is defined by the conditions (Def. 5).

(Def. 5)(i) The carrier of $\mathbb{N}_{\mathrm{Peano}} = \{0, 1\}$, and

(ii) for every symbol $x$ of $\mathbb{N}_{\mathrm{Peano}}$ and for every finite sequence $y$ of elements of the carrier of $\mathbb{N}_{\mathrm{Peano}}$ holds $x \Rightarrow y$ iff $x = 1$ but $y = \langle 0 \rangle$ or $y = \langle 1 \rangle$.

## 4. Properties of parse trees

Let $G$ be a non empty tree construction structure. We say that $G$ has terminals if and only if:

(Def. 6) The terminals of $G \neq \emptyset$.

We say that $G$ has nonterminals if and only if:

(Def. 7) The nonterminals of $G \neq \emptyset$.

We say that $G$ has useful nonterminals if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let $n_1$ be a symbol of $G$. Suppose $n_1 \in$ the nonterminals of $G$. Then there exists a finite sequence $p$ of elements of $\mathrm{TS}(G)$ such that $n_1 \Rightarrow$ the roots of $p$.

Let us note that there exists a non empty tree construction structure which is strict and has terminals, nonterminals, and useful nonterminals.

Let $G$ be a non empty tree construction structure with terminals. Then the terminals of $G$ is a non empty subset of $G$. Observe that $\mathrm{TS}(G)$ is non empty.

Let $G$ be a non empty tree construction structure with useful nonterminals. Note that $\mathrm{TS}(G)$ is non empty.

Let $G$ be a non empty tree construction structure with nonterminals. Then the nonterminals of $G$ is a non empty subset of $G$.

Let $G$ be a non empty tree construction structure with terminals. A terminal of $G$ is an element of the terminals of $G$.

Let $G$ be a non empty tree construction structure with nonterminals. A nonterminal of $G$ is an element of the nonterminals of $G$.

Let $G$ be a non empty tree construction structure with nonterminals and useful nonterminals and let $n_1$ be a nonterminal of $G$. A finite sequence of elements of $\mathrm{TS}(G)$ is said to be a subtree sequence joinable by $n_1$ if:

(Def. 9) $n_1 \Rightarrow$ the roots of it.

Let $G$ be a non empty tree construction structure with terminals and let $t$ be a terminal of $G$. Then the root tree of $t$ is an element of $\mathrm{TS}(G)$.

Let $G$ be a non empty tree construction structure with nonterminals and useful nonterminals, let $n_1$ be a nonterminal of $G$, and let $p$ be a subtree sequence joinable by $n_1$. Then $n_1$-tree$(p)$ is an element of $\mathrm{TS}(G)$.

We now state two propositions:

(9) Let $G$ be a non empty tree construction structure with terminals, $t_2$ be an element of $\mathrm{TS}(G)$, and $s$ be a terminal of $G$. If $t_2(\emptyset) = s$, then $t_2 =$ the root tree of $s$.

(10) Let $G$ be a non empty tree construction structure with terminals and nonterminals, $t_2$ be an element of $\mathrm{TS}(G)$, and $n_1$ be a nonterminal of $G$. Suppose $t_2(\emptyset) = n_1$. Then there exists a finite sequence $t_1$ of elements of $\mathrm{TS}(G)$ such that $t_2 = n_1\text{-tree}(t_1)$ and $n_1 \Rightarrow$ the roots of $t_1$.

## 5. THE EXAMPLE CONTINUED

Let us note that $\mathbb{N}_{\rm Peano}$ has terminals, nonterminals, and useful nonterminals.

Let $n_1$ be a nonterminal of $\mathbb{N}_{\rm Peano}$ and let $t$ be an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$. Then $n_1$-tree$(t)$ is an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$.

Let $x$ be a finite sequence of elements of $\mathbb{N}$. Let us assume that $x \neq \emptyset$. The functor $(x)({}_1{+}1)$ yielding a natural number is defined as follows:

(Def. 10) There exists a natural number $n$ such that $(x)({}_1{+}1) = n+1$ and $x(1) = n$.

The function $\mathbb{N}_{\rm Peano} \to \mathbb{N}$ from $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ into $\mathbb{N}$ is defined by the conditions (Def. 11).

(Def. 11)(i) For every symbol $t$ of $\mathbb{N}_{\rm Peano}$ such that $t \in$ the terminals of $\mathbb{N}_{\rm Peano}$ holds $(\mathbb{N}_{\rm Peano} \to \mathbb{N})$(the root tree of $t) = 0$, and

(ii) for every symbol $n_1$ of $\mathbb{N}_{\rm Peano}$ and for every finite sequence $t_1$ of elements of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ such that $n_1 \Rightarrow$ the roots of $t_1$ holds $(\mathbb{N}_{\rm Peano} \to \mathbb{N})(n_1\text{-tree}(t_1)) = ((\mathbb{N}_{\rm Peano} \to \mathbb{N}) \cdot t_1)({}_1{+}1)$.

Let $x$ be an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$. The functor $\mathrm{succ}(x)$ yields an element of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ and is defined by:

(Def. 12) $\mathrm{succ}(x) = 1\text{-tree}(\langle x \rangle)$.

The function $\mathbb{N} \to \mathbb{N}_{\rm Peano}$ from $\mathbb{N}$ into $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ is defined as follows:

(Def. 13) $(\mathbb{N} \to \mathbb{N}_{\rm Peano})(0) =$ the root tree of 0 and for every natural number $n$ holds $(\mathbb{N} \to \mathbb{N}_{\rm Peano})(n+1) = \mathrm{succ}((\mathbb{N} \to \mathbb{N}_{\rm Peano})(n))$.

The following propositions are true:

(11) For every element $p_1$ of $\mathrm{TS}(\mathbb{N}_{\rm Peano})$ holds $p_1 = (\mathbb{N} \to \mathbb{N}_{\rm Peano})((\mathbb{N}_{\rm Peano} \to \mathbb{N})(p_1))$.

(12) For every natural number $n$ holds $n = (\mathbb{N}_{\rm Peano} \to \mathbb{N})((\mathbb{N} \to \mathbb{N}_{\rm Peano})(n))$.

## 6. TREE TRAVERSALS AND TERMINAL LANGUAGE

Let $D$ be a set and let $F$ be a finite sequence of elements of $D^*$. The functor $\mathrm{Flat}(F)$ yielding an element of $D^*$ is defined as follows:

(Def. 14) There exists a binary operation $g$ on $D^*$ such that for all elements $p$, $q$ of $D^*$ holds $g(p, q) = p \frown q$ and $\mathrm{Flat}(F) = g \odot F$.

One can prove the following proposition

(13) For every set $D$ and for every element $d$ of $D^*$ holds $\mathrm{Flat}(\langle d \rangle) = d$.

Let $G$ be a non empty tree construction structure and let $t_2$ be a tree decorated with elements of the carrier of $G$. Let us assume that $t_2 \in \mathrm{TS}(G)$. The terminals of $t_2$ constitute a finite sequence of elements of the terminals of $G$ defined by the condition (Def. 15).

(Def. 15) There exists a function $f$ from $\mathrm{TS}(G)$ into (the terminals of $G)^*$ such that

(i) the terminals of $t_2 = f(t_2)$,

(ii) for every symbol $t$ of $G$ such that $t \in$ the terminals of $G$ holds $f$(the root tree of $t) = \langle t \rangle$, and

(iii) for every symbol $n_1$ of $G$ and for every finite sequence $t_1$ of elements of $\mathrm{TS}(G)$ such that $n_1 \Rightarrow$ the roots of $t_1$ holds $f(n_1\text{-tree}(t_1)) = \mathrm{Flat}(f \cdot t_1)$.

The pretraversal string of $t_2$ is a finite sequence of elements of the carrier of $G$ and is defined by the condition (Def. 16).

(Def. 16) There exists a function $f$ from $\mathrm{TS}(G)$ into (the carrier of $G)^*$ such that

(i) the pretraversal string of $t_2 = f(t_2)$,

(ii) for every symbol $t$ of $G$ such that $t \in$ the terminals of $G$ holds $f($the root tree of $t) = \langle t \rangle$, and

(iii) for every symbol $n_1$ of $G$ and for every finite sequence $t_1$ of elements of $\mathrm{TS}(G)$ and for every finite sequence $r_1$ such that $r_1 =$ the roots of $t_1$ and $n_1 \Rightarrow r_1$ and for every finite sequence $x$ of elements of (the carrier of $G)^*$ such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \langle n_1 \rangle \frown \mathrm{Flat}(x)$.

The posttraversal string of $t_2$ is a finite sequence of elements of the carrier of $G$ and is defined by the condition (Def. 17).

(Def. 17) There exists a function $f$ from $\mathrm{TS}(G)$ into (the carrier of $G)^*$ such that

(i) the posttraversal string of $t_2 = f(t_2)$,

(ii) for every symbol $t$ of $G$ such that $t \in$ the terminals of $G$ holds $f($the root tree of $t) = \langle t \rangle$, and

(iii) for every symbol $n_1$ of $G$ and for every finite sequence $t_1$ of elements of $\mathrm{TS}(G)$ and for every finite sequence $r_1$ such that $r_1 =$ the roots of $t_1$ and $n_1 \Rightarrow r_1$ and for every finite sequence $x$ of elements of (the carrier of $G)^*$ such that $x = f \cdot t_1$ holds $f(n_1\text{-tree}(t_1)) = \mathrm{Flat}(x) \frown \langle n_1 \rangle$.

Let $G$ be a non empty non empty tree construction structure with nonterminals and let $n_1$ be a symbol of $G$. The language derivable from $n_1$ is a subset of (the terminals of $G)^*$ and is defined by the condition (Def. 18).

(Def. 18) The language derivable from $n_1 = \{$the terminals of $t_2$; $t_2$ ranges over elements of FinTrees(the carrier of $G$): $t_2 \in \mathrm{TS}(G) \wedge t_2(\emptyset) = n_1\}$.

The language of pretraversals derivable from $n_1$ is a subset of (the carrier of $G)^*$ and is defined by the condition (Def. 19).

(Def. 19) The language of pretraversals derivable from $n_1 = \{$the pretraversal string of $t_2$; $t_2$ ranges over elements of FinTrees(the carrier of $G$): $t_2 \in \mathrm{TS}(G) \wedge t_2(\emptyset) = n_1\}$.

The language of posttraversals derivable from $n_1$ is a subset of (the carrier of $G)^*$ and is defined by the condition (Def. 20).

(Def. 20) The language of posttraversals derivable from $n_1 = \{$the posttraversal string of $t_2$; $t_2$ ranges over elements of FinTrees(the carrier of $G$): $t_2 \in \mathrm{TS}(G) \wedge t_2(\emptyset) = n_1\}$.

Next we state several propositions:

(14) For every tree $t$ decorated with elements of the carrier of $\mathbb{N}_{\mathrm{Peano}}$ such that $t \in \mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the terminals of $t = \langle 0 \rangle$.

(15) For every symbol $n_1$ of $\mathbb{N}_{\mathrm{Peano}}$ holds the language derivable from $n_1 = \{\langle 0 \rangle\}$.

(16) For every element $t$ of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the pretraversal string of $t = (\mathrm{height}\,\mathrm{dom}\,t \mapsto 1) \frown \langle 0 \rangle$.

(17) Let $n_1$ be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then

(i) if $n_1 = 0$, then the language of pretraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and

(ii) if $n_1 = 1$, then the language of pretraversals derivable from $n_1 = \{(n \mapsto 1) \frown \langle 0 \rangle; n$ ranges over natural numbers: $n \neq 0\}$.

(18) For every element $t$ of $\mathrm{TS}(\mathbb{N}_{\mathrm{Peano}})$ holds the posttraversal string of $t = \langle 0 \rangle \frown (\mathrm{height}\,\mathrm{dom}\,t \mapsto 1)$.

(19) Let $n_1$ be a symbol of $\mathbb{N}_{\mathrm{Peano}}$. Then

(i) if $n_1 = 0$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle\}$, and

(ii) if $n_1 = 1$, then the language of posttraversals derivable from $n_1 = \{\langle 0 \rangle \frown (n \mapsto 1); n$ ranges over natural numbers: $n \neq 0\}$.

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*Received October 12, 1993*

*Published January 2, 2004*