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Oriented Metric-Affine Plane — Part II

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Summary. Axiomatic description of properties of the oriented orthogonality relation. Next we construct (with the help of the oriented orthogonality relation) vector space and give the definitions of left-, right-, and semi-transitives.

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The articles [3], [2], [1], and [4] provide the notation and terminology for this paper.

In this paper V is a real linear space and x, y are vectors of V.

Let A_1 be a non empty affine structure and let a, b, c, d be elements of A_1. We introduce $a,b \top^{>} c,d$ as a synonym of $a,b \mathrel{\downharpoonleft\!\upharpoonright} c,d$.

One can prove the following propositions:

(1) Suppose x, y span the space. Then

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of $\mathrm{CESpace}(V,x,y)$ holds $u,u \top^{>} v,w$ and $u,v \top^{>} w,w$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} v_1,u_1$, then $u = v$ or $u_1 = v_1$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} u_1,w$, then $u,v \top^{>} v_1,w$ or $u,v \top^{>} w,v_1$ and if $u,v \top^{>} u_1,v_1$, then $v,u \top^{>} v_1,u_1$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} v_1,w$, then $u,v \top^{>} u_1,w$ and if $u,u_1 \top^{>} v,v_1$, then $v,v_1 \top^{>} u,u_1$ or $v,v_1 \top^{>} u_1,u$,

(ii) for all elements u, v, w of $\mathrm{CESpace}(V,x,y)$ there exists an element u_1 of $\mathrm{CESpace}(V,x,y)$ such that $w \neq u_1$ and $w,u_1 \top^{>} u,v$, and

(iii) for all elements u, v, w of $\mathrm{CESpace}(V,x,y)$ there exists an element u_1 of $\mathrm{CESpace}(V,x,y)$ such that $w \neq u_1$ and $u,v \top^{>} w,u_1$.

(2) Suppose x, y span the space. Then

(i) for all elements u, u_1, v, v_1, w, w_1, w_2 of $\mathrm{CMSpace}(V,x,y)$ holds $u,u \top^{>} v,w$ and $u,v \top^{>} w,w$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} v_1,u_1$, then $u = v$ or $u_1 = v_1$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} u_1,w$, then $u,v \top^{>} v_1,w$ or $u,v \top^{>} w,v_1$ and if $u,v \top^{>} u_1,v_1$, then $v,u \top^{>} v_1,u_1$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} v_1,w$, then $u,v \top^{>} u_1,w$ and if $u,u_1 \top^{>} v,v_1$, then $v,v_1 \top^{>} u,u_1$ or $v,v_1 \top^{>} u_1,u$,

(ii) for all elements u, v, w of $\mathrm{CMSpace}(V,x,y)$ there exists an element u_1 of $\mathrm{CMSpace}(V,x,y)$ such that $w \neq u_1$ and $w,u_1 \top^{>} u,v$, and

(iii) for all elements u, v, w of $\mathrm{CMSpace}(V,x,y)$ there exists an element u_1 of $\mathrm{CMSpace}(V,x,y)$ such that $w \neq u_1$ and $u,v \top^{>} w,u_1$.

Let I_1 be a non empty affine structure. We say that I_1 is oriented orthogonality if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For all elements u, u_1, v, v_1, w, w_1, w_2 of I_1 holds $u,u \top^{>} v,w$ and $u,v \top^{>} w,w$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} v_1,u_1$, then $u = v$ or $u_1 = v_1$ and if $u,v \top^{>} u_1,v_1$ and $u,v \top^{>} u_1,w$, then $u,v\top^{>}v_1,w$ or $u,v\top^{>}w,v_1$ and if $u,v\top^{>}u_1,v_1$, then $v,u\top^{>}v_1,u_1$ and if $u,v\top^{>}u_1,v_1$ and $u,v\top^{>}v_1,w$, then $u,v\top^{>}u_1,w$ and if $u,u_1\top^{>}v,v_1$, then $v,v_1\top^{>}u,u_1$ or $v,v_1\top^{>}u_1,u$,

(ii) for all elements u, v, w of I_1 there exists an element u_1 of I_1 such that $w \neq u_1$ and $w,u_1\top^{>}u,v$, and

(iii) for all elements u, v, w of I_1 there exists an element u_1 of I_1 such that $w \neq u_1$ and $u,v\top^{>}w,u_1$.

One can check that there exists a non empty affine structure which is oriented orthogonality.

An oriented orthogonality space is an oriented orthogonality non empty affine structure.

Next we state two propositions:

(4)[1] If x, y span the space, then $\mathrm{CMSpace}(V,x,y)$ is an oriented orthogonality space.

(5) If x, y span the space, then $\mathrm{CESpace}(V,x,y)$ is an oriented orthogonality space.

We use the following convention: A_1 is an oriented orthogonality space and u, u_1, u_2, v, v_1, v_2, w, w_1 are elements of A_1.

Next we state two propositions:

(6) For all elements u, v, w of A_1 there exists an element u_1 of A_1 such that $u_1,w\top^{>}u,v$ and $u_1 \neq w$.

(8)[2] For all elements u, v, w of A_1 there exists an element u_1 of A_1 such that $u \neq u_1$ but $v,w\top^{>}u,u_1$ or $v,w\top^{>}u_1,u$.

Let A_1 be an oriented orthogonality space and let a, b, c, d be elements of A_1. The predicate $a,b \perp c,d$ is defined by:

(Def. 2) $a,b\top^{>}c,d$ or $a,b\top^{>}d,c$.

Let A_1 be an oriented orthogonality space and let a, b, c, d be elements of A_1. The predicate $a,b \upharpoonleft\!\!\upharpoonright c,d$ is defined by:

(Def. 3) There exist elements e, f of A_1 such that $e \neq f$ and $e,f\top^{>}a,b$ and $e,f\top^{>}c,d$.

Let I_1 be an oriented orthogonality space. We say that I_1 is semi transitive if and only if:

(Def. 4) For all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of I_1 such that $u,u_1\top^{>}v,v_1$ and $w,w_1\top^{>}v,v_1$ and $w,w_1\top^{>}u_2,v_2$ holds $w = w_1$ or $v = v_1$ or $u,u_1\top^{>}u_2,v_2$.

Let I_1 be an oriented orthogonality space. We say that I_1 is right transitive if and only if:

(Def. 5) For all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of I_1 such that $u,u_1\top^{>}v,v_1$ and $v,v_1\top^{>}w,w_1$ and $u_2,v_2\top^{>}w,w_1$ holds $w = w_1$ or $v = v_1$ or $u,u_1\top^{>}u_2,v_2$.

Let I_1 be an oriented orthogonality space. We say that I_1 is left transitive if and only if:

(Def. 6) For all elements u, u_1, u_2, v, v_1, v_2, w, w_1 of I_1 such that $u,u_1\top^{>}v,v_1$ and $v,v_1\top^{>}w,w_1$ and $u,u_1\top^{>}u_2,v_2$ holds $u = u_1$ or $v = v_1$ or $u_2,v_2\top^{>}w,w_1$.

Let I_1 be an oriented orthogonality space. We say that I_1 is Euclidean like if and only if:

(Def. 7) For all elements u, u_1, v, v_1 of I_1 such that $u,u_1\top^{>}v,v_1$ holds $v,v_1\top^{>}u_1,u$.

Let I_1 be an oriented orthogonality space. We say that I_1 is Minkowskian like if and only if:

(Def. 8) For all elements u, u_1, v, v_1 of I_1 such that $u,u_1\top^{>}v,v_1$ holds $v,v_1\top^{>}u,u_1$.

One can prove the following propositions:

[1] The proposition (3) has been removed.
[2] The proposition (7) has been removed.

(9) $u,u_1 \upuparrows w,w$ and $w,w \upuparrows u,u_1$.

(10) If $u,u_1 \upuparrows v,v_1$, then $v,v_1 \upuparrows u,u_1$.

(11) If $u,u_1 \upuparrows v,v_1$, then $u_1,u \upuparrows v_1,v$.

(12) A_1 is left transitive iff for all v, v_1, w, w_1, u_2, v_2 such that $v,v_1 \upuparrows u_2,v_2$ and $v,v_1 \top^{>} w,w_1$ and $v \neq v_1$ holds $u_2,v_2 \top^{>} w,w_1$.

(13) A_1 is semi transitive iff for all u, u_1, u_2, v, v_1, v_2 such that $u,u_1 \top^{>} v,v_1$ and $v,v_1 \upuparrows u_2,v_2$ and $v \neq v_1$ holds $u,u_1 \top^{>} u_2,v_2$.

(14) If A_1 is semi transitive, then for all u, u_1, v, v_1, w, w_1 such that $u,u_1 \upuparrows v,v_1$ and $v,v_1 \upuparrows w,w_1$ and $v \neq v_1$ holds $u,u_1 \upuparrows w,w_1$.

(15) Suppose x, y span the space and $A_1 = \mathrm{CESpace}(V,x,y)$. Then A_1 is Euclidean like, left transitive, right transitive, and semi transitive.

Let us observe that there exists an oriented orthogonality space which is Euclidean like, left transitive, right transitive, and semi transitive.

Next we state the proposition

(16) Suppose x, y span the space and $A_1 = \mathrm{CMSpace}(V,x,y)$. Then A_1 is Minkowskian like, left transitive, right transitive, and semi transitive.

One can check that there exists an oriented orthogonality space which is Minkowskian like, left transitive, right transitive, and semi transitive.

One can prove the following propositions:

(17) If A_1 is left transitive, then A_1 is right transitive.

(18) If A_1 is left transitive, then A_1 is semi transitive.

(19) Suppose A_1 is semi transitive. Then A_1 is right transitive if and only if for all u, u_1, v, v_1, u_2, v_2 such that $u,u_1 \top^{>} u_2,v_2$ and $v,v_1 \top^{>} u_2,v_2$ and $u_2 \neq v_2$ holds $u,u_1 \upuparrows v,v_1$.

(20) If A_1 is right transitive, Euclidean like, and Minkowskian like, then A_1 is left transitive.

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