JOURNAL OF FORMALIZED MATHEMATICS
Volume 15, Released 2003, Published 2003
Inst. of Computer Science, Univ. of Białystok

# Convex Hull, Set of Convex Combinations and Convex Cone

Noboru Endou
Gifu National College of Technology

Yasunari Shidama
Shinshu University
Nagano

**Summary.** In this article, there are two themes. One of them is the proof that convex hull of a given subset $M$ consists of all convex combinations of $M$. Another is definitions of cone and convex cone and some properties of them.

MML Identifier: CONVEX3.

WWW: http://mizar.org/JFM/Vol15/convex3.html

The articles [7], [11], [8], [2], [12], [3], [5], [1], [4], [9], [10], and [6] provide the notation and terminology for this paper.

## 1. Equality of Convex Hull and Set of Convex Combinations

Let $V$ be a real linear space. The functor $\mathrm{ConvexComb}(V)$ yields a set and is defined by:

(Def. 1) For every set $L$ holds $L \in \mathrm{ConvexComb}(V)$ iff $L$ is a convex combination of $V$.

Let $V$ be a real linear space and let $M$ be a non empty subset of $V$. The functor $\mathrm{ConvexComb}(M)$ yields a set and is defined as follows:

(Def. 2) For every set $L$ holds $L \in \mathrm{ConvexComb}(M)$ iff $L$ is a convex combination of $M$.

We now state several propositions:

(1) Let $V$ be a real linear space and $v$ be a vector of $V$. Then there exists a convex combination $L$ of $V$ such that $\sum L = v$ and for every non empty subset $A$ of $V$ such that $v \in A$ holds $L$ is a convex combination of $A$.

(2) Let $V$ be a real linear space and $v_1$, $v_2$ be vectors of $V$. Suppose $v_1 \neq v_2$. Then there exists a convex combination $L$ of $V$ such that for every non empty subset $A$ of $V$ if $\{v_1, v_2\} \subseteq A$, then $L$ is a convex combination of $A$.

(3) Let $V$ be a real linear space and $v_1$, $v_2$, $v_3$ be vectors of $V$. Suppose $v_1 \neq v_2$ and $v_1 \neq v_3$ and $v_2 \neq v_3$. Then there exists a convex combination $L$ of $V$ such that for every non empty subset $A$ of $V$ if $\{v_1, v_2, v_3\} \subseteq A$, then $L$ is a convex combination of $A$.

(4) Let $V$ be a real linear space and $M$ be a non empty subset of $V$. Then $M$ is convex if and only if $\{\sum L; L$ ranges over convex combinations of $M$: $L \in \mathrm{ConvexComb}(V)\} \subseteq M$.

(5) Let $V$ be a real linear space and $M$ be a non empty subset of $V$. Then $\mathrm{conv}\, M = \{\sum L; L$ ranges over convex combinations of $M$: $L \in \mathrm{ConvexComb}(V)\}$.

## 2. Cone and Convex Cone

Let $V$ be a non empty RLS structure and let $M$ be a subset of $V$. We say that $M$ is cone if and only if:

(Def. 3) For every real number $r$ and for every vector $v$ of $V$ such that $r > 0$ and $v \in M$ holds $r \cdot v \in M$.

Next we state the proposition

(6) For every non empty RLS structure $V$ and for every subset $M$ of $V$ such that $M = \emptyset$ holds $M$ is cone.

Let $V$ be a non empty RLS structure. One can check that there exists a subset of $V$ which is cone.

Let $V$ be a non empty RLS structure. Observe that there exists a subset of $V$ which is empty and cone.

Let $V$ be a real linear space. Observe that there exists a subset of $V$ which is non empty and cone.

Next we state three propositions:

(7) Let $V$ be a non empty RLS structure and $M$ be a cone subset of $V$. Suppose $V$ is real linear space-like. Then $M$ is convex if and only if for all vectors $u$, $v$ of $V$ such that $u \in M$ and $v \in M$ holds $u + v \in M$.

(8) Let $V$ be a real linear space and $M$ be a subset of $V$. Then $M$ is convex and cone if and only if for every linear combination $L$ of $M$ such that the support of $L \neq \emptyset$ and for every vector $v$ of $V$ such that $v \in$ the support of $L$ holds $L(v) > 0$ holds $\sum L \in M$.

(9) For every non empty RLS structure $V$ and for all subsets $M$, $N$ of $V$ such that $M$ is cone and $N$ is cone holds $M \cap N$ is cone.

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*Received June 16, 2003*

*Published January 2, 2004*