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Components and Unions of Components

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Summary. First, we generalized **skl** function for a subset of topological spaces the value of which is the component including the set. Second, we introduced a concept of union of components a family of which has good algebraic properties. At the end, we discuss relationship between connectivity of a set as a subset in the whole space and as a subset of a subspace.

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The articles [5], [1], [3], [4], and [2] provide the notation and terminology for this paper.

1. The Component of a Subset in a Topological Space

In this paper G_1 is a non empty topological space.

Let G_1 be a topological structure and let V be a subset of G_1. The functor Component(V) yielding a subset of G_1 is defined by the condition (Def. 1).

(Def. 1) There exists a family F of subsets of G_1 such that for every subset A of G_1 holds $A \in F$ iff A is connected and $V \subseteq A$ and $\bigcup F =$ Component(V).

One can prove the following propositions:

(1) Let G_1 be a topological space and V be a subset of G_1. If there exists a subset A of G_1 such that A is connected and $V \subseteq A$, then $V \subseteq$ Component(V).

(2) Let G_1 be a topological space and V be a subset of G_1. If it is not true that there exists a subset A of G_1 such that A is connected and $V \subseteq A$, then Component$(V) = \emptyset$.

(3) Component$(\emptyset_{(G_1)}) =$ the carrier of G_1.

(4) For every subset V of G_1 such that V is connected holds Component$(V) \neq \emptyset$.

(5) For every topological space G_1 and for every subset V of G_1 such that V is connected and $V \neq \emptyset$ holds Component(V) is connected.

(6) For all subsets V, C of G_1 such that V is connected and C is connected holds if Component$(V) \subseteq C$, then $C =$ Component(V).

(7) For every subset A of G_1 such that A is a component of G_1 holds Component$(A) = A$.

(8) Let A be a subset of G_1. Then A is a component of G_1 if and only if there exists a subset V of G_1 such that V is connected and $V \neq \emptyset$ and $A =$ Component(V).

(9) For every subset V of G_1 such that V is connected and $V \neq \emptyset$ holds $\mathrm{Component}(V)$ is a component of G_1.

(10) For all subsets A, V of G_1 such that A is a component of G_1 and V is connected and $V \subseteq A$ and $V \neq \emptyset$ holds $A = \mathrm{Component}(V)$.

(11) For every subset V of G_1 such that V is connected and $V \neq \emptyset$ holds $\mathrm{Component}(\mathrm{Component}(V)) = \mathrm{Component}(V)$.

(12) For all subsets A, B of G_1 such that A is connected and B is connected and $A \neq \emptyset$ and $A \subseteq B$ holds $\mathrm{Component}(A) = \mathrm{Component}(B)$.

(13) For all subsets A, B of G_1 such that A is connected and B is connected and $A \neq \emptyset$ and $A \subseteq B$ holds $B \subseteq \mathrm{Component}(A)$.

(14) Let A be a subset of G_1 and B be a subset of G_1. If A is connected and $A \cup B$ is connected and $A \neq \emptyset$, then $A \cup B \subseteq \mathrm{Component}(A)$.

(15) For every subset A of G_1 and for every point p of G_1 such that A is connected and $p \in A$ holds $\mathrm{Component}(p) = \mathrm{Component}(A)$.

(16) Let A, B be subsets of G_1. Suppose A is connected and B is connected and A meets B. Then $A \cup B \subseteq \mathrm{Component}(A)$ and $A \cup B \subseteq \mathrm{Component}(B)$ and $A \subseteq \mathrm{Component}(B)$ and $B \subseteq \mathrm{Component}(A)$.

(17) For every subset A of G_1 such that A is connected and $A \neq \emptyset$ holds $\overline{A} \subseteq \mathrm{Component}(A)$.

(18) Let A, B be subsets of G_1. Suppose A is a component of G_1 and B is connected and $B \neq \emptyset$ and A misses B. Then A misses $\mathrm{Component}(B)$.

2. On Unions of Components

Let G_1 be a topological structure. A subset of G_1 is called a union of components of G_1 if it satisfies the condition (Def. 2).

(Def. 2) There exists a family F of subsets of G_1 such that for every subset B of G_1 such that $B \in F$ holds B is a component of G_1 and it $= \bigcup F$.

The following propositions are true:

(19) $\emptyset_{(G_1)}$ is a union of components of G_1.

(20) For every subset A of G_1 such that $A =$ the carrier of G_1 holds A is a union of components of G_1.

(21) Let A be a subset of G_1 and p be a point of G_1. If $p \in A$ and A is a union of components of G_1, then $\mathrm{Component}(p) \subseteq A$.

(22) Let A, B be subsets of G_1. Suppose A is a union of components of G_1 and B is a union of components of G_1. Then $A \cup B$ is a union of components of G_1 and $A \cap B$ is a union of components of G_1.

(23) Let F_1 be a family of subsets of G_1. Suppose that for every subset A of G_1 such that $A \in F_1$ holds A is a union of components of G_1. Then $\bigcup F_1$ is a union of components of G_1.

(24) Let F_1 be a family of subsets of G_1. Suppose that for every subset A of G_1 such that $A \in F_1$ holds A is a union of components of G_1. Then $\bigcap F_1$ is a union of components of G_1.

(25) Let A, B be subsets of G_1. Suppose A is a union of components of G_1 and B is a union of components of G_1. Then $A \setminus B$ is a union of components of G_1.

3. Operations Down and Up

Let G_1 be a topological structure, let B be a subset of G_1, and let p be a point of G_1. Let us assume that $p \in B$. The functor $\text{Down}(p,B)$ yields a point of $G_1 \restriction B$ and is defined as follows:

(Def. 3) $\text{Down}(p,B) = p$.

Let G_1 be a topological structure, let B be a subset of G_1, and let p be a point of $G_1 \restriction B$. Let us assume that $B \neq \emptyset$. The functor $\text{Up}(p)$ yielding a point of G_1 is defined as follows:

(Def. 4) $\text{Up}(p) = p$.

Let G_1 be a topological structure and let V, B be subsets of G_1. The functor $\text{Down}(V,B)$ yielding a subset of $G_1 \restriction B$ is defined by:

(Def. 5) $\text{Down}(V,B) = V \cap B$.

Let G_1 be a topological structure, let B be a subset of G_1, and let V be a subset of $G_1 \restriction B$. The functor $\text{Up}(V)$ yields a subset of G_1 and is defined by:

(Def. 6) $\text{Up}(V) = V$.

Let G_1 be a topological structure, let B be a subset of G_1, and let p be a point of G_1. Let us assume that $p \in B$. The functor $\text{skl}(p,B)$ yielding a subset of G_1 is defined as follows:

(Def. 7) For every point q of $G_1 \restriction B$ such that $q = p$ holds $\text{skl}(p,B) = \text{Component}(q)$.

The following propositions are true:

(26) For every subset B of G_1 and for every point p of G_1 such that $p \in B$ holds $\text{skl}(p,B) \neq \emptyset$.

(27) For every subset B of G_1 and for every point p of G_1 such that $p \in B$ holds $\text{skl}(p,B) = \text{Component}(\text{Down}(p,B))$.

(28) For all subsets V, B of G_1 such that $V \subseteq B$ holds $\text{Down}(V,B) = V$.

(29) For every topological space G_1 and for all subsets V, B of G_1 such that V is open holds $\text{Down}(V,B)$ is open.

(30) For all subsets V, B of G_1 such that $V \subseteq B$ holds $\overline{\text{Down}(V,B)} = \overline{V} \cap B$.

(31) For every subset B of G_1 and for every subset V of $G_1 \restriction B$ holds $\overline{V} = \overline{\text{Up}(V)} \cap B$.

(32) For all subsets V, B of G_1 such that $V \subseteq B$ holds $\overline{\text{Down}(V,B)} \subseteq \overline{V}$.

(33) For every subset B of G_1 and for every subset V of $G_1 \restriction B$ such that $V \subseteq B$ holds $\text{Down}(\text{Up}(V),B) = V$.

(34) Let G_1 be a topological space, V, B be subsets of G_1, and W be a subset of $G_1 \restriction B$. If $V = W$ and W is connected, then V is connected.

(35) For every subset B of G_1 and for every point p of G_1 such that $p \in B$ holds $\text{skl}(p,B)$ is connected.

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