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Introduction to Concept Lattices

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Summary. In this paper we give Mizar formalization of concept lattices. Concept lattices stem from the so-called formal concept analysis — a part of applied mathematics that brings mathematical methods into the field of data analysis and knowledge processing. Our approach follows the one given in [8].

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The articles [11], [6], [14], [12], [15], [4], [3], [17], [16], [5], [13], [9], [7], [1], [10], and [2] provide the notation and terminology for this paper.

1. FORMAL CONTEXTS

We introduce 2-sorted which are systems

⟨ objects, attributes ⟩,

where the objects and the attributes constitute sets.

Let C be a 2-sorted. We say that C is empty if and only if:

(Def. 1) The objects of C are empty and the attributes of C are empty.

We say that C is quasi-empty if and only if:

(Def. 2) The objects of C are empty or the attributes of C are empty.

Let us observe that there exists a 2-sorted which is strict and non empty and there exists a 2-sorted which is strict and non quasi-empty.

Let us observe that there exists a 2-sorted which is strict, empty, and quasi-empty.

We consider ContextStr as extensions of 2-sorted as systems

⟨ objects, attributes, an information ⟩,

where the objects and the attributes constitute sets and the information is a relation between the objects and the attributes.

Let us note that there exists a ContextStr which is strict and non empty and there exists a ContextStr which is strict and non quasi-empty.

A FormalContext is a non quasi-empty ContextStr.

Let C be a 2-sorted. An object of C is an element of the objects of C. An Attribute of C is an element of the attributes of C.

Let C be a non quasi-empty 2-sorted. One can check that the attributes of C is non empty and the objects of C is non empty.

Let C be a non quasi-empty 2-sorted. Observe that there exists a subset of the objects of C which is non empty and there exists a subset of the attributes of C which is non empty.

Let C be a FormalContext, let o be an object of C, and let a be an Attribute of C. We say that o is connected with a if and only if:

(Def. 5)[1] $\langle o, a\rangle \in$ the information of C.

We introduce o is not connected with a as an antonym of o is connected with a.

2. Derivation Operators

Let C be a FormalContext. The functor ObjectDerivation$C$ yields a function from $2^{\text{the objects of } C}$ into $2^{\text{the attributes of } C}$ and is defined by the condition (Def. 6).

(Def. 6) Let O be an element of $2^{\text{the objects of } C}$. Then $(\text{ObjectDerivation}C)(O) = \{a; a$ ranges over Attributes of C: $\bigwedge_{o:\text{object of } C} (o \in O \Rightarrow o$ is connected with $a)\}$.

Let C be a FormalContext. The functor AttributeDerivation$C$ yielding a function from $2^{\text{the attributes of } C}$ into $2^{\text{the objects of } C}$ is defined by the condition (Def. 7).

(Def. 7) Let A be an element of $2^{\text{the attributes of } C}$. Then $(\text{AttributeDerivation}C)(A) = \{o; o$ ranges over objects of C: $\bigwedge_{a:\text{Attribute of } C} (a \in A \Rightarrow o$ is connected with $a)\}$.

The following propositions are true:

(1) Let C be a FormalContext and o be an object of C. Then $(\text{ObjectDerivation}C)(\{o\}) = \{a; a$ ranges over Attributes of C: o is connected with $a\}$.

(2) Let C be a FormalContext and a be an Attribute of C. Then $(\text{AttributeDerivation}C)(\{a\}) = \{o; o$ ranges over objects of C: o is connected with $a\}$.

(3) For every FormalContext C and for all subsets O_1, O_2 of the objects of C such that $O_1 \subseteq O_2$ holds $(\text{ObjectDerivation}C)(O_2) \subseteq (\text{ObjectDerivation}C)(O_1)$.

(4) For every FormalContext C and for all subsets A_1, A_2 of the attributes of C such that $A_1 \subseteq A_2$ holds $(\text{AttributeDerivation}C)(A_2) \subseteq (\text{AttributeDerivation}C)(A_1)$.

(5) For every FormalContext C and for every subset O of the objects of C holds $O \subseteq (\text{AttributeDerivation}C)((\text{ObjectDerivation}C)(O))$.

(6) For every FormalContext C and for every subset A of the attributes of C holds $A \subseteq (\text{ObjectDerivation}C)((\text{AttributeDerivation}C)(A))$.

(7) For every FormalContext C and for every subset O of the objects of C holds $(\text{ObjectDerivation}C)(O) = (\text{ObjectDerivation}C)((\text{AttributeDerivation}C)((\text{ObjectDerivation}C)(O)))$.

(8) For every FormalContext C and for every subset A of the attributes of C holds $(\text{AttributeDerivation}C)(A) = (\text{AttributeDerivation}C)((\text{ObjectDerivation}C)((\text{AttributeDerivation}C)(A)))$.

(9) Let C be a FormalContext, O be a subset of the objects of C, and A be a subset of the attributes of C. Then $O \subseteq (\text{AttributeDerivation}C)(A)$ if and only if $[:O, A:] \subseteq$ the information of C.

(10) Let C be a FormalContext, O be a subset of the objects of C, and A be a subset of the attributes of C. Then $A \subseteq (\text{ObjectDerivation}C)(O)$ if and only if $[:O, A:] \subseteq$ the information of C.

(11) Let C be a FormalContext, O be a subset of the objects of C, and A be a subset of the attributes of C. Then $O \subseteq (\text{AttributeDerivation}C)(A)$ if and only if $A \subseteq (\text{ObjectDerivation}C)(O)$.

Let C be a FormalContext. The functor $\phi(C)$ yields a map from $2_{\subseteq}^{\text{the objects of } C}$ into $2_{\subseteq}^{\text{the attributes of } C}$ and is defined by:

(Def. 8) $\phi(C) = \text{ObjectDerivation}C$.

[1] The definitions (Def. 3) and (Def. 4) have been removed.

Let C be a FormalContext. The functor psi$C$ yields a map from $2_{\subseteq}^{\text{the attributes of } C}$ into $2_{\subseteq}^{\text{the objects of } C}$ and is defined as follows:

(Def. 9) psi$C$ = AttributeDerivation$C$.

Let P, R be non empty relational structures and let C_1 be a connection between P and R. We say that C_1 is co-Galois if and only if the condition (Def. 10) is satisfied.

(Def. 10) There exists a map f from P into R and there exists a map g from R into P such that

- (i) $C_1 = \langle f, g \rangle$,
- (ii) f is antitone,
- (iii) g is antitone, and
- (iv) for all elements p_1, p_2 of P and for all elements r_1, r_2 of R holds $p_1 \leq g(f(p_1))$ and $r_1 \leq f(g(r_1))$.

One can prove the following propositions:

(13)[2] Let P, R be non empty posets, C_1 be a connection between P and R, f be a map from P into R, and g be a map from R into P. Suppose $C_1 = \langle f, g \rangle$. Then C_1 is co-Galois if and only if for every element p of P and for every element r of R holds $p \leq g(r)$ iff $r \leq f(p)$.

(14) Let P, R be non empty posets and C_1 be a connection between P and R. Suppose C_1 is co-Galois. Let f be a map from P into R and g be a map from R into P. If $C_1 = \langle f, g \rangle$, then $f = f \cdot (g \cdot f)$ and $g = g \cdot (f \cdot g)$.

(15) For every FormalContext C holds $\langle \phi(C), \text{psi}C \rangle$ is co-Galois.

(16) For every FormalContext C and for all subsets O_1, O_2 of the objects of C holds $(\text{ObjectDerivation}C)(O_1 \cup O_2) = (\text{ObjectDerivation}C)(O_1) \cap (\text{ObjectDerivation}C)(O_2)$.

(17) For every FormalContext C and for all subsets A_1, A_2 of the attributes of C holds $(\text{AttributeDerivation}C)(A_1 \cup A_2) = (\text{AttributeDerivation}C)(A_1) \cap (\text{AttributeDerivation}C)(A_2)$.

(18) For every FormalContext C holds $(\text{ObjectDerivation}C)(\emptyset) =$ the attributes of C.

(19) For every FormalContext C holds $(\text{AttributeDerivation}C)(\emptyset) =$ the objects of C.

3. FORMAL CONCEPTS

Let C be a 2-sorted. We consider ConceptStr over C as systems

⟨ an extent, an intent ⟩,

where the extent is a subset of the objects of C and the intent is a subset of the attributes of C.

Let C be a 2-sorted and let C_2 be a ConceptStr over C. We say that C_2 is empty if and only if:

(Def. 11) The extent of C_2 is empty and the intent of C_2 is empty.

We say that C_2 is quasi-empty if and only if:

(Def. 12) The extent of C_2 is empty or the intent of C_2 is empty.

Let C be a non quasi-empty 2-sorted. One can check that there exists a ConceptStr over C which is strict and non empty and there exists a ConceptStr over C which is strict and quasi-empty.

Let C be an empty 2-sorted. One can check that every ConceptStr over C is empty.

Let C be a quasi-empty 2-sorted. One can check that every ConceptStr over C is quasi-empty.

Let C be a FormalContext and let C_2 be a ConceptStr over C. We say that C_2 is concept-like if and only if:

(Def. 13) $(\text{ObjectDerivation}C)$(the extent of C_2) = the intent of C_2 and $(\text{AttributeDerivation}C)$(the intent of C_2) = the extent of C_2.

[2] The proposition (12) has been removed.

Let C be a FormalContext. Observe that there exists a ConceptStr over C which is concept-like and non empty.

Let C be a FormalContext. A FormalConcept of C is a concept-like non empty ConceptStr over C.

Let C be a FormalContext. Observe that there exists a FormalConcept of C which is strict.

The following four propositions are true:

(20) Let C be a FormalContext and O be a subset of the objects of C. Then

(i) $\langle(\text{AttributeDerivation}C)((\text{ObjectDerivation}C)(O)),(\text{ObjectDerivation}C)(O)\rangle$ is a FormalConcept of C, and

(ii) for every subset O' of the objects of C and for every subset A' of the attributes of C such that $\langle O',A'\rangle$ is a FormalConcept of C and $O \subseteq O'$ holds $(\text{AttributeDerivation}C)((\text{ObjectDerivation}C)(O)) \subseteq O'$.

(21) Let C be a FormalContext and O be a subset of the objects of C. Then there exists a subset A of the attributes of C such that $\langle O,A\rangle$ is a FormalConcept of C if and only if $(\text{AttributeDerivation}C)((\text{ObjectDerivation}C)(O)) = O$.

(22) Let C be a FormalContext and A be a subset of the attributes of C. Then

(i) $\langle(\text{AttributeDerivation}C)(A),(\text{ObjectDerivation}C)((\text{AttributeDerivation}C)(A))\rangle$ is a FormalConcept of C, and

(ii) for every subset O' of the objects of C and for every subset A' of the attributes of C such that $\langle O',A'\rangle$ is a FormalConcept of C and $A \subseteq A'$ holds $(\text{ObjectDerivation}C)((\text{AttributeDerivation}C)(A)) \subseteq A'$.

(23) Let C be a FormalContext and A be a subset of the attributes of C. Then there exists a subset O of the objects of C such that $\langle O,A\rangle$ is a FormalConcept of C if and only if $(\text{ObjectDerivation}C)((\text{AttributeDerivation}C)(A)) = A$.

Let C be a FormalContext and let C_2 be a ConceptStr over C. We say that C_2 is universal if and only if:

(Def. 14) The extent of $C_2 =$ the objects of C.

Let C be a FormalContext and let C_2 be a ConceptStr over C. We say that C_2 is co-universal if and only if:

(Def. 15) The intent of $C_2 =$ the attributes of C.

Let C be a FormalContext. Note that there exists a FormalConcept of C which is strict and universal and there exists a FormalConcept of C which is strict and co-universal.

Let C be a FormalContext. The functor $\text{Concept}-\text{with}-\text{all}-\text{Objects}C$ yields a strict universal FormalConcept of C and is defined by the condition (Def. 16).

(Def. 16) There exists a subset O of the objects of C and there exists a subset A of the attributes of C such that $\text{Concept}-\text{with}-\text{all}-\text{Objects}C = \langle O,A\rangle$ and $O = (\text{AttributeDerivation}C)(\emptyset)$ and $A = (\text{ObjectDerivation}C)((\text{AttributeDerivation}C)(\emptyset))$.

Let C be a FormalContext. The functor $\text{Concept}-\text{with}-\text{all}-\text{Attributes}C$ yielding a strict co-universal FormalConcept of C is defined by the condition (Def. 17).

(Def. 17) There exists a subset O of the objects of C and there exists a subset A of the attributes of C such that $\text{Concept}-\text{with}-\text{all}-\text{Attributes}C = \langle O,A\rangle$ and $O = (\text{AttributeDerivation}C)((\text{ObjectDerivation}C)(\emptyset))$ and $A = (\text{ObjectDerivation}C)(\emptyset)$.

Next we state several propositions:

(24) Let C be a FormalContext. Then the extent of $\text{Concept}-\text{with}-\text{all}-\text{Objects}C =$ the objects of C and the intent of $\text{Concept}-\text{with}-\text{all}-\text{Attributes}C =$ the attributes of C.

(25) Let C be a FormalContext and C_2 be a FormalConcept of C. Then

(i) if the extent of $C_2 = \emptyset$, then C_2 is co-universal, and

(ii) if the intent of $C_2 = \emptyset$, then C_2 is universal.

(26) Let C be a FormalContext and C_2 be a strict FormalConcept of C. Then

(i) if the extent of $C_2 = \emptyset$, then $C_2 = \text{Concept} - \text{with} - \text{all} - \text{Attributes}C$, and

(ii) if the intent of $C_2 = \emptyset$, then $C_2 = \text{Concept} - \text{with} - \text{all} - \text{Objects}C$.

(27) Let C be a FormalContext and C_2 be a quasi-empty ConceptStr over C. Suppose C_2 is a FormalConcept of C. Then C_2 is universal and co-universal.

(28) Let C be a FormalContext and C_2 be a quasi-empty ConceptStr over C. If C_2 is a strict FormalConcept of C, then $C_2 = \text{Concept} - \text{with} - \text{all} - \text{Objects}C$ or $C_2 = \text{Concept} - \text{with} - \text{all} - \text{Attributes}C$.

Let C be a FormalContext. A non empty set is called a Set of FormalConcepts of C if:

(Def. 18) For every set X such that $X \in$ it holds X is a FormalConcept of C.

Let C be a FormalContext and let F_1 be a Set of FormalConcepts of C. We see that the element of F_1 is a FormalConcept of C.

Let C be a FormalContext and let C_3, C_4 be FormalConcepts of C. We say that C_3 is SubConcept of C_4 if and only if:

(Def. 19) The extent of $C_3 \subseteq$ the extent of C_4.

We introduce C_4 is SuperConcept of C_3 as a synonym of C_3 is SubConcept of C_4.

Next we state three propositions:

(31)[3] Let C be a FormalContext and C_3, C_4 be FormalConcepts of C. Then C_3 is SubConcept of C_4 if and only if the intent of $C_4 \subseteq$ the intent of C_3.

(33)[4] Let C be a FormalContext and C_3, C_4 be FormalConcepts of C. Then C_3 is SuperConcept of C_4 if and only if the intent of $C_3 \subseteq$ the intent of C_4.

(34) Let C be a FormalContext and C_2 be a FormalConcept of C. Then C_2 is SubConcept of $\text{Concept} - \text{with} - \text{all} - \text{Objects}C$ and $\text{Concept} - \text{with} - \text{all} - \text{Attributes}C$ is SubConcept of C_2.

4. CONCEPT LATTICES

Let C be a FormalContext. The functor $\text{B} - \text{carrier}C$ yields a non empty set and is defined by the condition (Def. 20).

(Def. 20) $\text{B} - \text{carrier}C = \{\langle E, I\rangle; E$ ranges over subsets of the objects of C, I ranges over subsets of the attributes of C: $\langle E, I\rangle$ is non empty $\wedge$ $(\text{ObjectDerivation}C)(E) = I$ $\wedge$ $(\text{AttributeDerivation}C)(I) = E\}$.

Let C be a FormalContext. Then $\text{B} - \text{carrier}C$ is a Set of FormalConcepts of C.

Let C be a FormalContext. Observe that $\text{B} - \text{carrier}C$ is non empty.

We now state the proposition

(35) For every FormalContext C and for every set C_2 holds $C_2 \in \text{B} - \text{carrier}C$ iff C_2 is a strict FormalConcept of C.

Let C be a FormalContext. The functor $\text{B} - \text{meet}C$ yields a binary operation on $\text{B} - \text{carrier}C$ and is defined by the condition (Def. 21).

[3] The propositions (29) and (30) have been removed.

[4] The proposition (32) has been removed.

(Def. 21) Let C_3, C_4 be strict FormalConcepts of C. Then there exists a subset O of the objects of C and there exists a subset A of the attributes of C such that $(\mathrm{B-meet}C)(C_3,C_4) = \langle O, A\rangle$ and $O =$ (the extent of C_3) $\cap$ (the extent of C_4) and $A = (\mathrm{ObjectDerivation}C)((\mathrm{AttributeDerivation}C)(($the intent of $C_3) \cup ($the intent of $C_4))))$.

Let C be a FormalContext. The functor $\mathrm{B-join}C$ yields a binary operation on $\mathrm{B-carrier}C$ and is defined by the condition (Def. 22).

(Def. 22) Let C_3, C_4 be strict FormalConcepts of C. Then there exists a subset O of the objects of C and there exists a subset A of the attributes of C such that $(\mathrm{B-join}C)(C_3,C_4) = \langle O, A\rangle$ and $O = (\mathrm{AttributeDerivation}C)((\mathrm{ObjectDerivation}C)(($the extent of $C_3) \cup ($the extent of $C_4)))$ and $A =$ (the intent of C_3) $\cap$ (the intent of C_4).

The following propositions are true:

(36) For every FormalContext C and for all strict FormalConcepts C_3, C_4 of C holds $(\mathrm{B-meet}C)(C_3,C_4) = (\mathrm{B-meet}C)(C_4,C_3)$.

(37) For every FormalContext C and for all strict FormalConcepts C_3, C_4 of C holds $(\mathrm{B-join}C)(C_3,C_4) = (\mathrm{B-join}C)(C_4,C_3)$.

(38) For every FormalContext C and for all strict FormalConcepts C_3, C_4, C_5 of C holds $(\mathrm{B-meet}C)(C_3, (\mathrm{B-meet}C)(C_4,C_5)) = (\mathrm{B-meet}C)((\mathrm{B-meet}C)(C_3,C_4),C_5)$.

(39) For every FormalContext C and for all strict FormalConcepts C_3, C_4, C_5 of C holds $(\mathrm{B-join}C)(C_3, (\mathrm{B-join}C)(C_4,C_5)) = (\mathrm{B-join}C)((\mathrm{B-join}C)(C_3,C_4),C_5)$.

(40) For every FormalContext C and for all strict FormalConcepts C_3, C_4 of C holds $(\mathrm{B-join}C)((\mathrm{B-meet}C)(C_3,C_4),C_4) = C_4$.

(41) For every FormalContext C and for all strict FormalConcepts C_3, C_4 of C holds $(\mathrm{B-meet}C)(C_3, (\mathrm{B-join}C)(C_3,C_4)) = C_3$.

(42) For every FormalContext C and for every strict FormalConcept C_2 of C holds $(\mathrm{B-meet}C)(C_2, \mathrm{Concept-with-all-Objects}C) = C_2$.

(43) For every FormalContext C and for every strict FormalConcept C_2 of C holds $(\mathrm{B-join}C)(C_2, \mathrm{Concept-with-all-Objects}C) = \mathrm{Concept-with-all-Objects}C$.

(44) For every FormalContext C and for every strict FormalConcept C_2 of C holds $(\mathrm{B-join}C)(C_2, \mathrm{Concept-with-all-Attributes}C) = C_2$.

(45) For every FormalContext C and for every strict FormalConcept C_2 of C holds $(\mathrm{B-meet}C)(C_2, \mathrm{Concept-with-all-Attributes}C) = \mathrm{Concept-with-all-Attributes}C$.

Let C be a FormalContext. The functor $\mathrm{ConceptLattice}C$ yielding a strict non empty lattice structure is defined as follows:

(Def. 23) $\mathrm{ConceptLattice}C = \langle \mathrm{B-carrier}C, \mathrm{B-join}C, \mathrm{B-meet}C\rangle$.

Next we state the proposition

(46) For every FormalContext C holds $\mathrm{ConceptLattice}C$ is a lattice.

Let C be a FormalContext. One can verify that $\mathrm{ConceptLattice}C$ is strict, non empty, and lattice-like.

Let C be a FormalContext and let S be a non empty subset of $\mathrm{ConceptLattice}C$. We see that the element of S is a FormalConcept of C.

Let C be a FormalContext and let C_2 be an element of $\mathrm{ConceptLattice}C$. The functor ${C_2}^{\mathrm{T}}$ yielding a strict FormalConcept of C is defined as follows:

(Def. 24) ${C_2}^{\mathrm{T}} = C_2$.

We now state two propositions:

(47) For every FormalContext C and for all elements C_3, C_4 of ConceptLattice$C$ holds $C_3 \sqsubseteq C_4$ iff $C_3{}^{\mathrm{T}}$ is SubConcept of $C_4{}^{\mathrm{T}}$.

(48) For every FormalContext C holds ConceptLattice$C$ is a complete lattice.

Let C be a FormalContext. One can verify that ConceptLattice$C$ is complete.

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