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Introduction to Banach and Hilbert Spaces — Part III

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Summary. The article is a continuation of [7] and of [8]. First we define the following concepts: the Cauchy sequence, the bounded sequence and the subsequence. The last part of this article contains definitions of the complete space and the Hilbert space.

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The articles [9], [2], [10], [1], [12], [3], [4], [5], [11], [6], [7], and [8] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: X is a real unitary space, x, g are points of X, a, r, M are real numbers, s_1, s_2, s_3, s_4 are sequences of X, N_1 is an increasing sequence of naturals, and k, n, m are natural numbers.

Let us consider X and let us consider s_1. We say that s_1 is Cauchy if and only if:

(Def. 1) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\rho(s_1(n), s_1(m)) < r$.

We introduce s_1 is a Cauchy sequence as a synonym of s_1 is Cauchy.

The following propositions are true:

(1) If s_1 is constant, then s_1 is a Cauchy sequence.

(2) s_1 is a Cauchy sequence if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|s_1(n) - s_1(m)\| < r$.

(3) If s_2 is a Cauchy sequence and s_3 is a Cauchy sequence, then $s_2 + s_3$ is a Cauchy sequence.

(4) If s_2 is a Cauchy sequence and s_3 is a Cauchy sequence, then $s_2 - s_3$ is a Cauchy sequence.

(5) If s_1 is a Cauchy sequence, then $a \cdot s_1$ is a Cauchy sequence.

(6) If s_1 is a Cauchy sequence, then $-s_1$ is a Cauchy sequence.

(7) If s_1 is a Cauchy sequence, then $s_1 + x$ is a Cauchy sequence.

(8) If s_1 is a Cauchy sequence, then $s_1 - x$ is a Cauchy sequence.

(9) If s_1 is convergent, then s_1 is a Cauchy sequence.

Let us consider X and let us consider s_2, s_3. We say that s_2 is compared to s_3 if and only if:

(Def. 2) For every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_2(n), s_3(n)) < r$.

The following propositions are true:

(10) s_1 is compared to s_1.

(11) If s_2 is compared to s_3, then s_3 is compared to s_2.

Let us consider X and let us consider s_2, s_3. Let us notice that the predicate s_2 is compared to s_3 is reflexive and symmetric.

Next we state several propositions:

(12) If s_2 is compared to s_3 and s_3 is compared to s_4, then s_2 is compared to s_4.

(13) s_2 is compared to s_3 iff for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_2(n) - s_3(n)\| < r$.

(14) If there exists k such that for every n such that $n \geq k$ holds $s_2(n) = s_3(n)$, then s_2 is compared to s_3.

(15) If s_2 is a Cauchy sequence and compared to s_3, then s_3 is a Cauchy sequence.

(16) If s_2 is convergent and compared to s_3, then s_3 is convergent.

(17) If s_2 is convergent and $\lim s_2 = g$ and s_2 is compared to s_3, then s_3 is convergent and $\lim s_3 = g$.

Let us consider X and let us consider s_1. We say that s_1 is bounded if and only if:

(Def. 3) There exists M such that $M > 0$ and for every n holds $\|s_1(n)\| \leq M$.

One can prove the following propositions:

(18) If s_2 is bounded and s_3 is bounded, then $s_2 + s_3$ is bounded.

(19) If s_1 is bounded, then $-s_1$ is bounded.

(20) If s_2 is bounded and s_3 is bounded, then $s_2 - s_3$ is bounded.

(21) If s_1 is bounded, then $a \cdot s_1$ is bounded.

(22) If s_1 is constant, then s_1 is bounded.

(23) For every m there exists M such that $M > 0$ and for every n such that $n \leq m$ holds $\|s_1(n)\| < M$.

(24) If s_1 is convergent, then s_1 is bounded.

(25) If s_2 is bounded and compared to s_3, then s_3 is bounded.

Let us consider X, N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of X.

Let X be a non empty 1-sorted structure and let s_5, s be sequences of X. We say that s_5 is a subsequence of s if and only if:

(Def. 4) There exists an increasing sequence N of naturals such that $s_5 = s \cdot N$.

One can prove the following propositions:

(26) Let X be a real unitary space, s be a sequence of X, N be an increasing sequence of naturals, and n be a natural number. Then $(s \cdot N)(n) = s(N(n))$.

(27) s_1 is a subsequence of s_1.

(28) If s_2 is a subsequence of s_3 and s_3 is a subsequence of s_4, then s_2 is a subsequence of s_4.

(29) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(30) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

(31) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(32) If s_1 is convergent and s_2 is a subsequence of s_1, then s_2 is convergent.

(33) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(34) If s_1 is a Cauchy sequence and s_2 is a subsequence of s_1, then s_2 is a Cauchy sequence.

Let us consider X, let us consider s_1, and let us consider k. The functor $s_1 \uparrow k$ yielding a sequence of X is defined as follows:

(Def. 5) For every n holds $(s_1 \uparrow k)(n) = s_1(n+k)$.

Next we state a number of propositions:

(35) $s_1 \uparrow 0 = s_1$.

(36) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(37) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k+m)$.

(38) $(s_2 + s_3) \uparrow k = s_2 \uparrow k + s_3 \uparrow k$.

(39) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(40) $(s_2 - s_3) \uparrow k = s_2 \uparrow k - s_3 \uparrow k$.

(41) $(a \cdot s_1) \uparrow k = a \cdot (s_1 \uparrow k)$.

(42) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(43) $s_1 \uparrow k$ is a subsequence of s_1.

(44) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(46)[1] If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(47) If s_1 is a Cauchy sequence and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is a Cauchy sequence.

(48) If s_1 is a Cauchy sequence, then $s_1 \uparrow k$ is a Cauchy sequence.

(49) If s_2 is compared to s_3, then $s_2 \uparrow k$ is compared to $s_3 \uparrow k$.

(50) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(51) If s_1 is constant, then $s_1 \uparrow k$ is constant.

Let us consider X. We say that X is complete if and only if:

(Def. 6) For every s_1 such that s_1 is a Cauchy sequence holds s_1 is convergent.

We introduce X is a complete space as a synonym of X is complete.

One can prove the following proposition

(53)[2] If X is a complete space and s_1 is a Cauchy sequence, then s_1 is bounded.

Let us consider X. We say that X is Hilbert if and only if:

(Def. 7) X is a real unitary space and a complete space.

We introduce X is a Hilbert space as a synonym of X is Hilbert.

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[1] The proposition (45) has been removed.

[2] The proposition (52) has been removed.

REFERENCES

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/nat_1.html.

[2] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/ordinal1.html.

[3] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[4] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[5] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/seqm_3.html.

[6] Jan Popiołek. Real normed space. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/normsp_1.html.

[7] Jan Popiołek. Introduction to Banach and Hilbert spaces — part I. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/bhsp_1.html.

[8] Jan Popiołek. Introduction to Banach and Hilbert spaces — part II. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/bhsp_2.html.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[10] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/rlvect_1.html.

[12] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

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