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On the Group of Automorphisms of Universal Algebra and Many Sorted Algebra

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Summary. The aim of the article is to check the compatibility of the automorphisms of universal algebras introduced in [10] and the corresponding concept for many sorted algebras introduced in [11].

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The articles [15], [8], [21], [22], [5], [7], [6], [2], [4], [16], [17], [13], [19], [20], [1], [12], [3], [10], [14], [18], [11], and [9] provide the notation and terminology for this paper.

1. On the Group of Automorphisms of Universal Algebra

In this paper U_1 is a universal algebra and f, g are functions from U_1 into U_1.

We now state the proposition

(1) $\mathrm{id}_{\text{the carrier of } U_1}$ is an isomorphism of U_1 and U_1.

Let us consider U_1. The functor $\mathrm{UAAut}(U_1)$ yields a non empty set of functions from the carrier of U_1 to the carrier of U_1 and is defined by the conditions (Def. 1).

(Def. 1)(i) Every element of $\mathrm{UAAut}(U_1)$ is a function from U_1 into U_1, and

(ii) for every function h from U_1 into U_1 holds $h \in \mathrm{UAAut}(U_1)$ iff h is an isomorphism of U_1 and U_1.

Next we state several propositions:

(2) $\mathrm{UAAut}(U_1) \subseteq (\text{the carrier of } U_1)^{\text{the carrier of } U_1}$.

(4)[1] $\mathrm{id}_{\text{the carrier of } U_1} \in \mathrm{UAAut}(U_1)$.

(5) For all f, g such that f is an element of $\mathrm{UAAut}(U_1)$ and $g = f^{-1}$ holds g is an isomorphism of U_1 and U_1.

(6) For every element f of $\mathrm{UAAut}(U_1)$ holds $f^{-1} \in \mathrm{UAAut}(U_1)$.

[1] The proposition (3) has been removed.

(7) For all elements f_1, f_2 of $\mathrm{UAAut}(U_1)$ holds $f_1 \cdot f_2 \in \mathrm{UAAut}(U_1)$.

Let us consider U_1. The functor $\mathrm{UAAutComp}(U_1)$ yielding a binary operation on $\mathrm{UAAut}(U_1)$ is defined by:

(Def. 2) For all elements x, y of $\mathrm{UAAut}(U_1)$ holds $(\mathrm{UAAutComp}(U_1))(x,y) = y \cdot x$.

Let us consider U_1. The functor $\mathrm{UAAutGroup}(U_1)$ yields a group and is defined by:

(Def. 3) $\mathrm{UAAutGroup}(U_1) = \langle \mathrm{UAAut}(U_1), \mathrm{UAAutComp}(U_1) \rangle$.

Let us consider U_1. One can check that $\mathrm{UAAutGroup}(U_1)$ is strict.

The following propositions are true:

(8) For all elements x, y of $\mathrm{UAAutGroup}(U_1)$ and for all elements f, g of $\mathrm{UAAut}(U_1)$ such that $x = f$ and $y = g$ holds $x \cdot y = g \cdot f$.

(9) $\mathrm{id}_{\text{the carrier of } U_1} = 1_{\mathrm{UAAutGroup}(U_1)}$.

(10) For every element f of $\mathrm{UAAut}(U_1)$ and for every element g of $\mathrm{UAAutGroup}(U_1)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

2. Some Properties of Many Sorted Functions

In the sequel I denotes a set and A, B, C denote many sorted sets indexed by I.

Let us consider I, A, B. We say that A is transformable to B if and only if:

(Def. 4) For every set i such that $i \in I$ holds if $B(i) = \emptyset$, then $A(i) = \emptyset$.

Let us note that the predicate A is transformable to B is reflexive.

Next we state several propositions:

(11) If A is transformable to B and B is transformable to C, then A is transformable to C.

(12) For every set x and for every many sorted set A indexed by $\{x\}$ holds $A = \{x\} \longmapsto A(x)$.

(13) For all function yielding functions F, G, H holds $(H \circ G) \circ F = H \circ (G \circ F)$.

(14) Let A, B be non-empty many sorted sets indexed by I and F be a many sorted function from A into B. If F is “1-1” and onto, then F^{-1} is “1-1” and onto.

(15) Let A, B be non-empty many sorted sets indexed by I and F be a many sorted function from A into B. If F is “1-1” and onto, then $(F^{-1})^{-1} = F$.

(16) For all function yielding functions F, G such that F is “1-1” and G is “1-1” holds $G \circ F$ is “1-1”.

(17) Let B, C be non-empty many sorted sets indexed by I, F be a many sorted function from A into B, and G be a many sorted function from B into C. If F is onto and G is onto, then $G \circ F$ is onto.

(18) Let A, B, C be non-empty many sorted sets indexed by I, F be a many sorted function from A into B, and G be a many sorted function from B into C. If F is “1-1” and onto and G is “1-1” and onto, then $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$.

(19) Let A, B be non-empty many sorted sets indexed by I, F be a many sorted function from A into B, and G be a many sorted function from B into A. If F is “1-1” and onto and $G \circ F = \mathrm{id}_A$, then $G = F^{-1}$.

3. On the Group of Automorphisms of Many Sorted Algebra

In the sequel S denotes a non void non empty many sorted signature and U_2, U_3 denote non-empty algebras over S.

Let us consider I, A, B. The functor $\mathrm{MSFuncs}(A,B)$ yielding a many sorted set indexed by I is defined by:

(Def. 5) For every set i such that $i \in I$ holds $(\mathrm{MSFuncs}(A,B))(i) = B(i)^{A(i)}$.

One can prove the following three propositions:

(21)[2] Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let x be a set. If $x \in \prod \mathrm{MSFuncs}(A,B)$, then x is a many sorted function from A into B.

(22) Let A, B be many sorted sets indexed by I. Suppose A is transformable to B. Let g be a many sorted function from A into B. Then $g \in \prod \mathrm{MSFuncs}(A,B)$.

(23) For all many sorted sets A, B indexed by I such that A is transformable to B holds $\mathrm{MSFuncs}(A,B)$ is non-empty.

Let us consider I, A, B. Let us assume that A is transformable to B. A non empty set is called a set of many sorted functions from A into B if:

(Def. 6) For every set x such that $x \in$ it holds x is a many sorted function from A into B.

Let us consider I, A. One can verify that $\mathrm{MSFuncs}(A,A)$ is non-empty.

Let us consider S, U_2, U_3. A set of many sorted functions from U_2 into U_3 is a set of many sorted functions from the sorts of U_2 into the sorts of U_3.

Let I be a set and let D be a many sorted set indexed by I. Note that there exists a set of many sorted functions from D into D which is non empty.

Let I be a set, let D be a many sorted set indexed by I, and let A be a non empty set of many sorted functions from D into D. We see that the element of A is a many sorted function from D into D.

The following propositions are true:

(24) id_A is onto.

(25) id_A is "1-1".

(27)[3] $\mathrm{id}_{\text{the sorts of } U_2} \in \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

Let us consider S, U_2. The functor $\mathrm{MSAAut}(U_2)$ yields a set of many sorted functions from the sorts of U_2 into the sorts of U_2 and is defined by the conditions (Def. 7).

(Def. 7)(i) Every element of $\mathrm{MSAAut}(U_2)$ is a many sorted function from U_2 into U_2, and

(ii) for every many sorted function h from U_2 into U_2 holds $h \in \mathrm{MSAAut}(U_2)$ iff h is an isomorphism of U_2 and U_2.

Next we state several propositions:

(29)[4] For every element f of $\mathrm{MSAAut}(U_2)$ holds $f \in \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(30) $\mathrm{MSAAut}(U_2) \subseteq \prod \mathrm{MSFuncs}(\text{the sorts of } U_2, \text{the sorts of } U_2)$.

(31) $\mathrm{id}_{\text{the sorts of } U_2} \in \mathrm{MSAAut}(U_2)$.

(32) For every element f of $\mathrm{MSAAut}(U_2)$ holds $f^{-1} \in \mathrm{MSAAut}(U_2)$.

[2] The proposition (20) has been removed.
[3] The proposition (26) has been removed.
[4] The proposition (28) has been removed.

(33) For all elements f_1, f_2 of $\mathrm{MSAAut}(U_2)$ holds $f_1 \circ f_2 \in \mathrm{MSAAut}(U_2)$.

(34) For every many sorted function F from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$ and for every element f of $\mathrm{UAAut}(U_1)$ such that $F = \{0\} \longmapsto f$ holds $F \in \mathrm{MSAAut}(\mathrm{MSAlg}(U_1))$.

Let us consider S, U_2. The functor $\mathrm{MSAAutComp}(U_2)$ yields a binary operation on $\mathrm{MSAAut}(U_2)$ and is defined by:

(Def. 8) For all elements x, y of $\mathrm{MSAAut}(U_2)$ holds $(\mathrm{MSAAutComp}(U_2))(x,y) = y \circ x$.

Let us consider S, U_2. The functor $\mathrm{MSAAutGroup}(U_2)$ yielding a group is defined by:

(Def. 9) $\mathrm{MSAAutGroup}(U_2) = \langle \mathrm{MSAAut}(U_2), \mathrm{MSAAutComp}(U_2) \rangle$.

Let us consider S, U_2. Observe that $\mathrm{MSAAutGroup}(U_2)$ is strict.
The following propositions are true:

(35) For all elements x, y of $\mathrm{MSAAutGroup}(U_2)$ and for all elements f, g of $\mathrm{MSAAut}(U_2)$ such that $x = f$ and $y = g$ holds $x \cdot y = g \circ f$.

(36) $\mathrm{id}_{\text{the sorts of } U_2} = 1_{\mathrm{MSAAutGroup}(U_2)}$.

(37) For every element f of $\mathrm{MSAAut}(U_2)$ and for every element g of $\mathrm{MSAAutGroup}(U_2)$ such that $f = g$ holds $f^{-1} = g^{-1}$.

4. On the Relationship of Automorphisms of 1-sorted and ManySorted Algebras

The following propositions are true:

(38) Let U_4, U_5 be universal algebras. Suppose U_4 and U_5 are similar. Let F be a many sorted function from $\mathrm{MSAlg}(U_4)$ into $(\mathrm{MSAlg}(U_5)\,\mathrm{over}\,\mathrm{MSSign}(U_4))$. Then $F(0)$ is a function from U_4 into U_5.

(39) For every element f of $\mathrm{UAAut}(U_1)$ holds $\{0\} \longmapsto f$ is a many sorted function from $\mathrm{MSAlg}(U_1)$ into $\mathrm{MSAlg}(U_1)$.

(40) Let h be a function. Suppose $\mathrm{dom}\, h = \mathrm{UAAut}(U_1)$ and for every set x such that $x \in \mathrm{UAAut}(U_1)$ holds $h(x) = \{0\} \longmapsto x$. Then h is a homomorphism from $\mathrm{UAAutGroup}(U_1)$ to $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$.

(41) Let h be a homomorphism from $\mathrm{UAAutGroup}(U_1)$ to $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$. If for every set x such that $x \in \mathrm{UAAut}(U_1)$ holds $h(x) = \{0\} \longmapsto x$, then h is an isomorphism.

(42) $\mathrm{UAAutGroup}(U_1)$ and $\mathrm{MSAAutGroup}(\mathrm{MSAlg}(U_1))$ are isomorphic.

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