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A Projective Closure and Projective Horizon of an Affine Space

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Summary. With every affine space A we correlate two incidence structures. The first, called Inc-ProjSp(A), is the usual projective closure of A, i.e. the structure obtained from A by adding directions of lines and planes of A. The second, called projective horizon of A, is the structure built from directions. We prove that Inc-ProjSp(A) is always a projective space, and projective horizon of A is a projective space provided A is at least 3-dimensional. Some evident relationships between projective and affine configurational axioms that may hold in A and in Inc-ProjSp(A) are established.

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The articles [9], [2], [11], [8], [13], [12], [14], [1], [5], [6], [7], [3], [10], and [4] provide the notation and terminology for this paper.

We follow the rules: A_1 denotes an affine space, A, K, M, X, Y denote subsets of A_1, and x, y denote sets.

Next we state several propositions:

(1) If A_1 is an affine plane and $X =$ the carrier of A_1, then X is a plane.

(2) If A_1 is an affine plane and X is a plane, then $X =$ the carrier of A_1.

(3) If A_1 is an affine plane and X is a plane and Y is a plane, then $X = Y$.

(4) If $X =$ the carrier of A_1 and X is a plane, then A_1 is an affine plane.

(5) If A not // K and $A || X$ and $A || Y$ and $K || X$ and $K || Y$ and A is a line and K is a line and X is a plane and Y is a plane, then $X || Y$.

(6) If X is a plane and $A || X$ and $X || Y$, then $A || Y$.

Let us consider A_1. The lines of A_1 yielding a family of subsets of A_1 is defined as follows:

(Def. 1) The lines of $A_1 = \{A : A \text{ is a line}\}$.

Let us consider A_1. The planes of A_1 yielding a family of subsets of A_1 is defined as follows:

(Def. 2) The planes of $A_1 = \{A : A \text{ is a plane}\}$.

The following propositions are true:

(7) For every x holds $x \in$ the lines of A_1 iff there exists X such that $x = X$ and X is a line.

(8) For every x holds $x \in$ the planes of A_1 iff there exists X such that $x = X$ and X is a plane.

Let us consider A_1. The parallelity of lines of A_1 yielding an equivalence relation of the lines of A_1 is defined by:

(Def. 3) The parallelity of lines of $A_1 = \{\langle K, M\rangle : K$ is a line $\wedge$ M is a line $\wedge$ $K||M\}$.

Let us consider A_1. The parallelity of planes of A_1 yielding an equivalence relation of the planes of A_1 is defined by:

(Def. 4) The parallelity of planes of $A_1 = \{\langle X, Y\rangle : X$ is a plane $\wedge$ Y is a plane $\wedge$ $X||Y\}$.

Let us consider A_1, X. Let us assume that X is a line. The direction of X yields a subset of the lines of A_1 and is defined by:

(Def. 5) The direction of $X = [X]_{\text{the parallelity of lines of } A_1}$.

Let us consider A_1, X. Let us assume that X is a plane. The direction of X yielding a subset of the planes of A_1 is defined as follows:

(Def. 6) The direction of $X = [X]_{\text{the parallelity of planes of } A_1}$.

We now state several propositions:

(9) If X is a line, then for every x holds $x \in$ the direction of X iff there exists Y such that $x = Y$ and Y is a line and $X||Y$.

(10) If X is a plane, then for every x holds $x \in$ the direction of X iff there exists Y such that $x = Y$ and Y is a plane and $X||Y$.

(11) If X is a line and Y is a line, then the direction of $X =$ the direction of Y iff X // Y.

(12) If X is a line and Y is a line, then the direction of $X =$ the direction of Y iff $X||Y$.

(13) If X is a plane and Y is a plane, then the direction of $X =$ the direction of Y iff $X||Y$.

Let us consider A_1. The directions of lines of A_1 yielding a non empty set is defined as follows:

(Def. 7) The directions of lines of $A_1 =$ Classes(the parallelity of lines of A_1).

Let us consider A_1. The directions of planes of A_1 yielding a non empty set is defined as follows:

(Def. 8) The directions of planes of $A_1 =$ Classes(the parallelity of planes of A_1).

One can prove the following propositions:

(14) For every x holds $x \in$ the directions of lines of A_1 iff there exists X such that $x =$ the direction of X and X is a line.

(15) Let given x. Then $x \in$ the directions of planes of A_1 if and only if there exists X such that $x =$ the direction of X and X is a plane.

(16) The carrier of A_1 misses the directions of lines of A_1.

(17) If A_1 is an affine plane, then the lines of A_1 misses the directions of planes of A_1.

(18) For every x holds $x \in$ [: the lines of A_1, $\{1\}$:] iff there exists X such that $x = \langle X, 1\rangle$ and X is a line.

(19) Let given x. Then $x \in$ [: the directions of planes of A_1, $\{2\}$:] if and only if there exists X such that $x = \langle$the direction of X, $2\rangle$ and X is a plane.

Let us consider A_1. The projective points over A_1 yielding a non empty set is defined by:

(Def. 9) The projective points over $A_1 =$ (the carrier of A_1) $\cup$ (the directions of lines of A_1).

Let us consider A_1. The functor $L(A_1)$ yields a non empty set and is defined as follows:

(Def. 10) $L(A_1) = [:$ the lines of A_1, $\{1\}:] \cup [:$ the directions of planes of A_1, $\{2\}:]$.

Let us consider A_1. The functor $\mathbf{I}_{(A_1)}$ yielding a relation between the projective points over A_1 and $L(A_1)$ is defined by the condition (Def. 11).

(Def. 11) Let given x, y. Then $\langle x, y\rangle \in \mathbf{I}_{(A_1)}$ if and only if one of the following conditions is satisfied:

(i) there exists K such that K is a line but $y = \langle K, 1\rangle$ but $x \in$ the carrier of A_1 and $x \in K$ or $x =$ the direction of K, or

(ii) there exist K, X such that K is a line and X is a plane and $x =$ the direction of K and $y = \langle$the direction of X, $2\rangle$ and $K || X$.

Let us consider A_1. The incidence of directions of A_1 yielding a relation between the directions of lines of A_1 and the directions of planes of A_1 is defined by the condition (Def. 12).

(Def. 12) Let given x, y. Then $\langle x, y\rangle \in$ the incidence of directions of A_1 if and only if there exist A, X such that $x =$ the direction of A and $y =$ the direction of X and A is a line and X is a plane and $A || X$.

Let us consider A_1. The functor Inc-ProjSp(A_1) yielding a strict projective incidence structure is defined by:

(Def. 13) Inc-ProjSp$(A_1) = \langle$the projective points over A_1, $L(A_1), \mathbf{I}_{(A_1)}\rangle$.

Let us consider A_1. The projective horizon of A_1 yields a strict projective incidence structure and is defined by the condition (Def. 14).

(Def. 14) The projective horizon of $A_1 = \langle$the directions of lines of A_1, the directions of planes of A_1, the incidence of directions of $A_1\rangle$.

Next we state several propositions:

(20) Let given x. Then x is a point of Inc-ProjSp(A_1) if and only if one of the following conditions is satisfied:

(i) x is an element of A_1, or

(ii) there exists X such that $x =$ the direction of X and X is a line.

(21) x is an element of the points of the projective horizon of A_1 if and only if there exists X such that $x =$ the direction of X and X is a line.

(22) If x is an element of the points of the projective horizon of A_1, then x is a point of Inc-ProjSp(A_1).

(23) Let given x. Then x is a line of Inc-ProjSp(A_1) if and only if there exists X such that $x = \langle X, 1\rangle$ and X is a line or $x = \langle$the direction of X, $2\rangle$ and X is a plane.

(24) x is an element of the lines of the projective horizon of A_1 if and only if there exists X such that $x =$ the direction of X and X is a plane.

(25) If x is an element of the lines of the projective horizon of A_1, then $\langle x, 2\rangle$ is a line of Inc-ProjSp(A_1).

For simplicity, we use the following convention: x denotes an element of A_1, X, Y, X' denote subsets of A_1, a, p, q denote points of Inc-ProjSp(A_1), and A denotes a line of Inc-ProjSp(A_1).

The following propositions are true:

(26) If $x = a$ and $\langle X, 1\rangle = A$, then a lies on A iff X is a line and $x \in X$.

(27) If $x = a$ and $\langle$the direction of X, $2\rangle = A$ and X is a plane, then a does not lie on A.

(28) If $a =$ the direction of Y and $\langle X, 1\rangle = A$ and Y is a line and X is a line, then a lies on A iff $Y || X$.

(29) Suppose $a =$ the direction of Y and $A = \langle$the direction of $X,\ 2\rangle$ and Y is a line and X is a plane. Then a lies on A if and only if $Y || X$.

(30) If X is a line and $a =$ the direction of X and $A = \langle X, 1\rangle$, then a lies on A.

(31) Suppose X is a line and Y is a plane and $X \subseteq Y$ and $a =$ the direction of X and $A = \langle$the direction of $Y,\ 2\rangle$. Then a lies on A.

(32) Suppose Y is a plane and $X \subseteq Y$ and $X' \mathbin{//} X$ and $a =$ the direction of X' and $A = \langle$the direction of $Y,\ 2\rangle$. Then a lies on A.

(33) If $A = \langle$the direction of $X,\ 2\rangle$ and X is a plane and a lies on A, then a is not an element of A_1.

(34) If $A = \langle X, 1\rangle$ and X is a line and p lies on A and p is not an element of A_1, then $p =$ the direction of X.

(35) Suppose $A = \langle X, 1\rangle$ and X is a line and p lies on A and a lies on A and $a \neq p$ and p is not an element of the carrier of A_1. Then a is an element of A_1.

(36) Let a be an element of the points of the projective horizon of A_1 and A be an element of the lines of the projective horizon of A_1. Suppose $a =$ the direction of X and $A =$ the direction of Y and X is a line and Y is a plane. Then a lies on A if and only if $X || Y$.

(37) Let a be an element of the points of the projective horizon of A_1, a' be an element of the points of Inc-ProjSp(A_1), A be an element of the lines of the projective horizon of A_1, and A' be a line of Inc-ProjSp(A_1). If $a' = a$ and $A' = \langle A, 2\rangle$, then a lies on A iff a' lies on A'.

In the sequel P, Q are lines of Inc-ProjSp(A_1).
Next we state several propositions:

(38) Let a, b be elements of the points of the projective horizon of A_1 and A, K be elements of the lines of the projective horizon of A_1. Suppose a lies on A and a lies on K and b lies on A and b lies on K. Then $a = b$ or $A = K$.

(39) Let A be an element of the lines of the projective horizon of A_1. Then there exist elements a, b, c of the points of the projective horizon of A_1 such that a lies on A and b lies on A and c lies on A and $a \neq b$ and $b \neq c$ and $c \neq a$.

(40) Let a, b be elements of the points of the projective horizon of A_1. Then there exists an element A of the lines of the projective horizon of A_1 such that a lies on A and b lies on A.

(41) Let x, y be elements of the points of the projective horizon of A_1 and X be an element of the lines of Inc-ProjSp(A_1). Suppose $x \neq y$ and $\langle x, X\rangle \in$ the incidence of Inc-ProjSp(A_1) and $\langle y, X\rangle \in$ the incidence of Inc-ProjSp(A_1). Then there exists an element Y of the lines of the projective horizon of A_1 such that $X = \langle Y, 2\rangle$.

(42) Let x be a point of Inc-ProjSp(A_1) and X be an element of the lines of the projective horizon of A_1. Suppose $\langle x, \langle X, 2\rangle\rangle \in$ the incidence of Inc-ProjSp(A_1). Then x is an element of the points of the projective horizon of A_1.

(43) Suppose Y is a plane and X is a line and X' is a line and $X \subseteq Y$ and $X' \subseteq Y$ and $P = \langle X, 1\rangle$ and $Q = \langle X', 1\rangle$. Then there exists q such that q lies on P and q lies on Q.

(44) Let a, b, c, d, p be elements of the points of the projective horizon of A_1 and M, N, P, Q be elements of the lines of the projective horizon of A_1. Suppose that a lies on M and b lies on M and c lies on N and d lies on N and p lies on M and p lies on N and a lies on P and c lies on P and b lies on Q and d lies on Q and p does not lie on P and p does not lie on Q and $M \neq N$. Then there exists an element q of the points of the projective horizon of A_1 such that q lies on P and q lies on Q.

Let us consider A_1. Note that Inc-ProjSp(A_1) is partial, linear, at least 2-dimensional, at least 3-rank, and Vebleian.

Let us observe that there exists a projective space defined in terms of incidence which is strict and 2-dimensional.

Let A_1 be an affine plane. Observe that Inc-ProjSp(A_1) is 2-dimensional.

The following propositions are true:

(45) If Inc-ProjSp(A_1) is 2-dimensional, then A_1 is an affine plane.

(46) Suppose A_1 is not an affine plane. Then the projective horizon of A_1 is a projective space defined in terms of incidence.

(47) Suppose the projective horizon of A_1 is a projective space defined in terms of incidence. Then A_1 is not an affine plane.

(48) Let M, N be subsets of A_1 and o, a, b, c, a', b', c' be elements of the carrier of A_1. Suppose that M is a line and N is a line and $M \neq N$ and $o \in M$ and $o \in N$ and $o \neq a$ and $o \neq a'$ and $o \neq b$ and $o \neq b'$ and $o \neq c$ and $o \neq c'$ and $a \in M$ and $b \in M$ and $c \in M$ and $a' \in N$ and $b' \in N$ and $c' \in N$ and $a, b' \upharpoonleft\upharpoonright b, a'$ and $b, c' \upharpoonleft\upharpoonright c, b'$ and $a = b$ or $b = c$ or $a = c$. Then $a, c' \upharpoonleft\upharpoonright c, a'$.

(49) If Inc-ProjSp(A_1) is Pappian, then A_1 is Pappian.

(50) Let A, P, C be subsets of A_1 and o, a, b, c, a', b', c' be elements of the carrier of A_1. Suppose that $o \in A$ and $o \in P$ and $o \in C$ and $o \neq a$ and $o \neq b$ and $o \neq c$ and $a \in A$ and $a' \in A$ and $b \in P$ and $b' \in P$ and $c \in C$ and $c' \in C$ and A is a line and P is a line and C is a line and $A \neq P$ and $A \neq C$ and $a, b \upharpoonleft\upharpoonright a', b'$ and $a, c \upharpoonleft\upharpoonright a', c'$ and $o = a'$ or $a = a'$. Then $b, c \upharpoonleft\upharpoonright b', c'$.

(51) If Inc-ProjSp(A_1) is Desarguesian, then A_1 is Desarguesian.

(52) If Inc-ProjSp(A_1) is Fanoian, then A_1 is Fanoian.

REFERENCES

[1] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[2] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[3] Wojciech Leończuk, Henryk Oryszczyszyn, and Krzysztof Prażmowski. Planes in affine spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/aff_4.html.

[4] Wojciech Leończuk and Krzysztof Prażmowski. Incidence projective spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/incproj.html.

[5] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/analoaf.html.

[6] Henryk Oryszczyszyn and Krzysztof Prażmowski. Ordered affine spaces defined in terms of directed parallelity — part I. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/diraf.html.

[7] Henryk Oryszczyszyn and Krzysztof Prażmowski. Parallelity and lines in affine spaces. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/aff_1.html.

[8] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/eqrel_1.html.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[10] Wojciech A. Trybulec. Axioms of incidency. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/incsp_1.html.

[11] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[12] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[13] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[14] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

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